Numerical Analysis Exam
May 12, 1999

Problem 1

Consider the floating point system $F(\beta, t, L, U)$ with $\beta = 10, t = 4, L = -2, U = 3$.
a) Give an example that illustrates that the following identity might not hold when we use finite precision calculations: $A * (B - C) = A * B - A * C$.
b) Give an example of an operation that will result in an *overflow* error message.
c) Give an example of an operation that will result in an *underflow* error message.

Problem 2

Let A be a non-singular $n \times n$ matrix. Consider a rank-1 update of A:

$$\bar{A} = A - uv^T$$

where u, v are n-vectors.

The Sherman-Morrison formula gives an expression for the inverse of $\bar{A}$ in terms of A^{-1} as follows:

$$\bar{A}^{-1} = A^{-1} + \alpha(A^{-1}u)(v^T A^{-1})$$

where $\alpha = 1/(1 - v^T A^{-1} u)$.
a) Use the Sherman-Morrison formula to outline an algorithm to solve a linear system of the form $\bar{A}x = b$ without forming $\bar{A}$.
b) Explain how the algorithm in (a) can be efficiently used in the case when we have a program that can solve linear systems involving A and A^T. Discuss the arithmetic cost of this algorithm.
c) Suppose that A is a symmetric positive definite tridiagonal matrix. What is the arithmetic complexity of solving the system $\bar{A}x = b$ using the algorithm in (a)? What is the arithmetic complexity of solving the system $\bar{A}x = b$ by first computing the elements of $\bar{A}$ and then applying Gaussian elimination?

Problem 3

Given the function $f(x) = 1 - x^2$ on $[0, 1]$, we consider the problem of approximating $f(x)$ by a continuous piece-wise linear polynomial $p(x)$ with nodes at $x_0 = 0, x_1 = 0.5$ and $x_2 = 1$, with the condition that $p(1) = 0$.
a) Find a basis for the subspace just described. What is its dimension?
b) Find the least-squares approximation to f in this subspace.
c) What is the error in the least squares norm? In the maximum norm?

Problem 4

a) Show that the quadrature formula:

$$I(p) = w_1 p(x_1) + w_2 p(x_2) + w_3 p(x_3)$$

cannot be exact for $\int_{-1}^{1} p(x)dx$ for all polynomials p of degree m, if m is greater than 5, regardless of the choice of w_1, w_2, w_3, x_1, x_2 and x_3.
Hint: Given a choice of abscissas x_1, x_2, x_3, define a polynomial P_6 of degree 6 such that $P_6(x) \leq 0$ for all $x \in [0, 1]$ and $P_6(x_i) = 0, i = 1, 2, 3$.

b) Describe the choice of weights w_j and the nodes x_j $(j = 1, 2, 3)$ such that the formula is exact for all polynomials of degree $m \leq 5$. What is the name of this quadrature rule?

Problem 5

Compute the Lagrange interpolating polynomial of degree 2 with nodes 0, 2, 5 for the function $f(x) = (1 + x)^{-2}$. What is the relative error of your approximation at $x = 4$?

Problem 6

True or False: The fixed point iteration $x_{n+1} = g(x_n)$ where $g(x) = (x + 5)/(x + 1)$, converges linearly but not quadratically to the square root of 5 (justify your answer).

True or False: The fixed point iteration $x_{n+1} = g(x_n)$ where $g(x) = \frac{1}{2}(x + \frac{5}{x})$ converges quadratically to the square root of 5 (justify your answer).

Problem 7

Explain the two main sources of error in numerical differentiation.