Comprehensive Exam: Numerical Analysis

Date: May 16

Time: 3 hours

Problem 1

In solving the quadratic equation $x^2 - 10^5 x + 1 = 0$, the use of the familiar formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

will cause a problem. Investigate the example, observe the difficulty, and propose a remedy.

Problem 2

Suppose that the values of $f(x) = 1 - \cos(x)$ are required near $x = 0$. A careless programmer may write the Fortran statement function

$$\mathbf{F(X) = 1 - COS(X)} \tag{2}$$

not realizing that serious loss of accuracy will occur. Investigate the example, observe the difficulty, and propose a remedy.

Problem 3

The Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad (-\infty < x < \infty) \tag{3}$$

is well known. How many terms of the above expansion does one need to compute to relative accuracy 10^{-6}

a) e^{1}
b) e^{10}
c) e^{-1}
d) e^{-10}

Investigate the example, observe the difficulty (if there is one), and propose a remedy.

Problem 4

The well-known Schulz method for the inversion of matrices is an iterative scheme

$$X_{k+1} = 2 \cdot X_k - X_k \cdot A \cdot X_k, \tag{4}$$

where A is the matrix to be inverted, and X_k is the approximation obtained on the k-th step. Prove that the scheme (4) is quadratically convergent.

Problem 5

Prove that the order of convergence of the numerical differentiation formula

$$f'(x) \sim \frac{8 \cdot (f(x+h) - f(x-h)) - (f(x+2 \cdot h) - f(x-2 \cdot h))}{12 \cdot h} \tag{5}$$

is equal to 4.

Problem 6

Determine the condition number of the 3×3-matrix

$$\begin{matrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 9 \end{matrix} \tag{6}$$

Problem 7

Suppose that a, b are a pair of real numbers such that $a < b$, and that $n \geq 2$ is an integer. For a function $f : [a, b] \to R^1$, the n point trapezoidal rule $T_n(f)$ is defined by the formula

$$T_n(f) = h\left(\sum_{i=0}^{n-1} f(a + ih) - \left(\frac{f(a) + f(b)}{2}\right)\right), \tag{7}$$

with

$$h = (b - a)/(n - 1) \tag{8}$$

Consider the use of the trapezoidal rule to evaluate the integral

$$\int_0^{2\pi} f(x)dx, \tag{9}$$

for each of the following functions

a) $f(x) = \cos(11.1x) + \sin(11.2x)$,

b) $f(x) = e^{-x}$,

c) $f(x) = \cos(11x) + \sin(12x)$,

d) $f(x) = \sqrt{|x|}$,

e) $f(x) = e^{-x^2}$

What convergence rate do you expect for each of them?