Comprehensive Exam: Numerical Analysis

Date: May 22 2001

Time: 3 hours

Problem 1

In solving the quadratic equation $x^2 - 10^5 x + 1 = 0$, the use of the familiar quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

will cause a problem. Investigate the example, observe the difficulty, and propose a remedy.

Problem 2

Suppose that the values of $f(x) = x - \sin(x)$ are required near $x = 0$. The careless programmer may write the Fortran statement function

$$\mathbf{F(X) = X - SIN(X)} \tag{2}$$

not realizing that serious loss of accuracy will occur. Investigate the example, observe the difficulty, and propose a remedy.

Problem 3

Derive a recursive formula to calculate the cube root of a positive number a using Newton's method. Suggest an appropriate starting point

Problem 4

Find the Lagrange polynomial that approximates $f(x) = x^3$, using the nodes $x_0 = -1, x_1 = 0, x_2 = 1$.

Problem 5

A central difference formula for the approximation of the second derivative is given by the formula

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \quad (3)$$

Show that it is second order.

Problem 6

Prove that the inverse of a non-singular upper-triangular matrix is an upper-triangular matrix.

Problem 7

Suppose that a, b are a pair of real numbers such that $a < b$, and that $n \geq 2$ is an integer. For a function $f : [a, b] \to R^1$, the n-point trapezoidal rule $T_n(f)$ is defined by the formula

$$T_n(f) = h\left(\frac{f(a)}{2} + f(a+h) + \ldots + f(b-h) + \frac{f(b)}{2}\right) \quad (4)$$

with

$$h = (b-a)/(n-1). \quad (5)$$

Consider the use of the trapezoidal rule to evaluate the integral

$$\int_0^{2\pi} f(x)dx, \quad (6)$$

for each of the following functions. What convergence rate do you expect from each of them?

a) $f(x) = e^{-x}$,

b) $f(x) = \cos(11x) + \sin(12x)$,

c) $f(x) = \sqrt{x}$,

d) $f(x) = x^3 \log x$.