Comprehensive Exam
Scientific Computing
19 May 2003

All problems have equal weight. You may consult Mathews and Fink, *Numerical Methods Using Matlab*.

1. Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. The quadratic formula states that the roots are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Describe four distinct instances where evaluating these formulae in floating-point arithmetic would give significantly less accurate answers than the underlying precision might lead one to expect. How could you compute the roots more accurately in each case?

2. Consider the system of linear equations

$$\begin{aligned} 34x + 55y &= 21 \\ 55x + 89y &= 34. \end{aligned}$$

Two approximate solutions are $x_1 = -0.11$, $y_1 = 0.45$ and $x_2 = -0.99$, $y_2 = 1.01$. Explain how each can be considered to be superior to the other numerically.

3. Let $A = (a_1, a_2, \ldots, a_n)$ be an $n \times n$ matrix whose LU decomposition is given, and let

$$B_i = (a_1, a_2, \ldots, a_{i-1}, g, a_{i+1}, \ldots, a_n).$$

Give $O(n^2)$ algorithms for solving the linear system $B_i x = d$ and for computing the LU decomposition of B_n.

4. a) Let $f(x)$ be an odd function (i.e., $f(x) = -f(-x)$ for all x) so that $f(0) = 0$. Assume that $f'(0) \neq 0$ so that Newton's method will converge to the root $x = 0$ of $f(x) = 0$ given a sufficiently accurate initial guess. Prove that the order of convergence is three, i.e., that Newon's method is cubically convergent for this problem.

   b) Let $f(x)$ be a strictly convex function (i.e., $f''(x) > 0$ for all x) and assume that $x = 0$ is the only root of $f(x) = 0$. Prove that Newton's method will converge given *any* initial guess.

5. a) For which values of α does there exist a unique polynomial $p_\alpha(x)$ of degree 2 or less such that $p_\alpha(0) = 0$, $p_\alpha(1) = 1$, and $p'_\alpha(\alpha) = 2$?

   b) For which values of α does there exist a unique polynomial $q_\alpha(x)$ of degree 4 or less such that $q_\alpha(0) = 0$, $q'_\alpha(0) = 0$, $q_\alpha(1) = 1$, $q'_\alpha(1) = 1$, and $q''_\alpha(\alpha) = 2$?

6. A Gaussian quadrature is one of the form

$$Q_n(f) \equiv \sum_{i=1}^{n} w_i f(x_i) \approx \int_0^1 f(x)dx$$

that is exact for the monomials 1, x, x^2, $\ldots$, x^{2n-2}, and x^{2n-1}. Prove that the weights w_i must be nonnegative. Hint: The integral of a nonnegative function is nonnegative.