

Comprehensive Exam: Numerical Analysis

Date: May 18

Time: 3 hours; you may use a calculator and one book

Problem 1

Determine the interpolating polynomial approximating the function on the interval $[-2, 3]$, given the data below. Express the polynomial in the form $a_0 + a_1x + a_2x^2 + \dots$.

x	-2	-1	0	1	2	3
$p(x)$	-5	1	1	1	7	25

Problem 2

Suppose that we wish to use linear interpolation to approximate $\log_e(x)$ for all $x \in [1, 2]$ to an accuracy 10^{-6} from a table of values at equispaced points. Determine the largest spacing for which this is possible.

Problem 3

Two famous numerical analysts, Doofus and Smartypants, are asked to find the root closest to 0 of the polynomial

$$80 - 116x + 60x^2 - 13x^3 + x^4. \quad (1)$$

Doofus implements Newton's method, comes up with the sequence of iterations depicted in Table 1 below, and proclaims that the iteration is quadratic. Is Doofus right? What kind of convergence is exhibited in Table 1?

Meanwhile, Smartypants observes Doofus' implementation and decides that he can modify the Newton in order to obtain a higher order of convergence. He obtains Table 2. How, and more importantly, why did Smartypants have to modify Newton's method in order to obtain better convergence to the root?

Table 1: Doofus

1	0.6896551724E+00
2	0.1174998031E+01
3	0.1503982576E+01
4	0.1715901111E+01
5	0.1844400978E+01
6	0.1917745107E+01
7	0.1957555056E+01
8	0.1978415265E+01
9	0.1989112319E+01
10	0.1994531690E+01

Table2: Smartypants

1	0.1379310345E+01
2	0.1894775372E+01
3	0.1995765052E+01
4	0.1999992554E+01
5	0.2000000000E+01
6	0.2000000000E+01
7	0.2000000000E+01
8	0.2000000000E+01
9	0.2000000000E+01
10	0.2000000000E+01

Problem 4

Find the spectrum (all eigenvalues) of the 3×3 -matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Problem 5

Suppose that A is a non-singular $n \times n$ -matrix, and $U, V \in \mathbb{R}^n$ are two (column) vectors. Show that

$$(A - UV^*)^{-1} = A^{-1} + \alpha A^{-1}UV^*A^{-1}, \quad (2)$$

with

$$\alpha = \frac{1}{1 - V^* A^{-1} U}. \quad (3)$$

Under what conditions does the inverse (2) exist?

Problem 6

Determine the condition number of the 3×3 -matrix

$$\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 9 \end{array} \quad (4)$$