Comprehensive Examination in Theory

May 19, 2004

Question 1: (40 points = 8+8+8+8+8)

Answer each of the following in 5 lines or less:

(a) Is there a set $L \subseteq \{0,1\}^*$ with the property that every subset of $\{0,1\}^*$ is many-one reducible to it? Explain.

(b) Argue briefly that every finite set is regular.

(c) Suppose G is a CFG generating a CFL L. Suppose I delete a production from G to get a new CFG G'. Argue that $L(G') \subseteq L(G)$.

(d) You are given a matrix A with real entries. The length of a column of A is its Euclidean length—the square root of the sum of the squares of the entries in the column. Describe a polynomial time algorithm that will find a subset I of columns of A so that

  (i) I is a linearly independent set of columns, and

  (ii) the product of the lengths of the columns in I is maximized subject to (i).

(e) Suppose a directed graph $G(V, E)$ has non-negative edge lengths. We are to find the length of the shortest path from s to every vertex in G. Suppose we are already told the length of the shortest path from s to some of the vertices. How would we incorporate this information into Dijkstra's algorithm so as to avoid recomputing this information?

Question 2: (12 points)

Suppose $G(V, E)$ is a directed graph with two special vertices, s and t. A set of s–t directed paths is said to be *edge disjoint* if each edge of G is used in at most one of the paths in the set. Argue that the maximum cardinality of a set of edge-disjoint s–t paths is equal to the maximum s–t flow when all edge capacities are set to 1. Use this to prove Menger's Theorem—the maximum number of edge-disjoint s–t paths equals the minimum number of edges whose removal disconnects s from t.

Question 3: (20 points = 15 +5)

There are n objects: $1, 2, \ldots n$. We are to pack a subset of these objects into a knapsack. For $i = 1, 2, \ldots n$, we are given the weight a_i, the volume b_i, and benefit c_i of object i, all non-negative integers. You are also given the maximum total weight w and the maximum total volume v of objects that can be packed into the knapsack.

(a) Describe an algorithm which runs in time bounded by a polynomial in $n, \sum_i c_i, \sum_i b_i, \sum_i a_i$ and produces the subset of objects maximizing the total benefit while meeting the total weight and volume restrictions.

(b) Briefly discuss whether the above algorithm can be called a polynomial time bounded algorithm.

Question 4: (28 points = 14 +14)

Prove that the following languages are NP complete :

(a) The set of graphs $G(V, E)$ with the property that G either has a Hamiltonian cycle or a clique of size $|V|/2$.

(b) The set of $n+1$-tuples of integers $(a_1, a_2, \ldots a_n, b)$ such that $a_1, a_2, \ldots a_n$ are all odd integers and there is a subset $I \subseteq \{1, 2, \ldots n\}$ such that $\sum_{i \in I} a_i = b$.