# Nash Bargaining via Flexible Budget Markets

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In his seminal 1950 paper, John Nash defined the bargaining game; the ensuing theory of bargaining lies today at the heart of game theory. In this paper, we initiate a study of Nash bargaining games via combinatorial, polynomial time algorithms (i.e., algorithms that conduct an efficient search over a discrete space). We also carry this program over to solving nonsymmetric bargaining games of Kalai (1977).

We define the class LNB of games in which the set of feasible utility vectors of agents is a polytope specified by finitely many packing constraints – the Nash bargaining solution of each such game is captured as the optimal solution to a convex program that has only linear constraints. We show that several natural games in this class always have rational solutions and we give primal-dual, combinatorial algorithms for solving them. We also indicate how the primal-dual paradigm, which was normally run in the framework of linear programs, can be enhanced to solving these nonlinear programs.

The combinatorial nature of our algorithms, the structural insights gained in obtaining them, and our classification of games within LNB (obtained by imposing restrictions on the convex programs capturing their solutions), has led to novel insights into game-theoretic properties of the solution concepts of Nash and nonsymmetric bargaining games, see Chakrabarty et. al. (2009). Additionally, we believe that the rationality and combinatorial solvability of convex programs studied in Devanur et. al. (*JACM* 2008), Jain & Vazirani (*GEB*, to appear) and the current paper appear to have identified the tip of an iceberg and the way to move forward is to seek combinatorial approximation algorithms for solving specific classes of convex programs.

## ADNB, flexible budget markets, and enhancing the primal-dual paradigm

Our main algorithmic result pertains to a Nash bargaining game derived from the linear case of the Arrow-Debreu market model: Given linear utility functions of $n$ agents over $g$ divisible goods and their disagreement utilities (which, in particular, may be the utilities they derive from their initial endowments), find a way of distributing the goods according to the Nash bargaining solution. We call this game **ADNB**.

For solving **ADNB** (and other games in LNB), we first reduce it to a new market model, which we call a *flexible budget market*. This new model is a natural variant of the linear case of the classic model given by Irwing Fisher in 1891: Given linear utility functions of $n$ agents over $g$ divisible goods, the money possessed by each agent and the amount of each good available, find equilibrium prices, i.e., prices at which supply equals demand and the market clears. Whereas in Fisher's model, each buyer wants to derive the maximum possible utility by spending a fixed amount of money, in our model, each buyer wants to derive a stated amount of utility by buying goods in the most cost-effective manner; clearly, the money spent by each buyer will be a function of the announced prices of goods, hence requiring buyers to have flexible budgets. Unlike Fisher's model, which always has an equilibrium, ours may not.

In our primal-dual algorithm, allocations of goods are the primal variables and prices are dual variables. As with almost all primal-dual algorithms, our algorithm raises dual variable greedily – it starts with very low prices and systematically raises them until the surplus money of buyers vanishes and equilibrium is found. However, the more complex nature of KKT conditions raises new difficulties in making this overall scheme work out.

The fundamental difference between complimentary slackness conditions for linear programs and KKT conditions for convex programs is that whereas the former do not involve both primal and dual variables simultaneously in an equality constraint, the latter do. So, despite our simple dual growth process, primal objects go tight and loose, hence making the design of the algorithm and its analysis complex.

The following 2 new difficulties arise over and above Fisher's linear case (Devanur et. al.):
1). As we increase prices of goods, the money of buyers changes, and so the surplus money of some buyers may actually increase.
2). We need to determine if the given market has an equilibrium.

We define the new notion of *good* buyer – one whose surplus money decreases as prices are raised. Our algorithm consists of two stages: Stage I checks for feasibility, i.e., existence of equilibirum, and terminates with prices such that either all buyers are rendered good or a proof of infeasibility is found; the latter being a suitable dual solution to an LP that checks for feasibility. In the former case, Stage II finds the equilibrium by raising prices in a systematic manner. In both stages, the addition difficulties of the enhanced setup are overcome by using $l_2$ norm in defining the potential function that accounts for progress and the use of the notion of balanced flows from Devanur et. al. (*JACM* 2008). We also solve one of their open questions by giving a family of examples to show that $l_2$ norm cannot be dispensed with.

Another intriguing aspect of our algorithm is that whereas Stage II operates by raising dual variables, Stage I actually needs to lower them. The only other primal-dual algorithm we know of that raises and lowers duals is Edmonds' algorithm for maximum weighted matching.

## Subclasses of LNB

In bargaining theory, 2-person games occupy a special place. It turns out that these games occupy a special place algorithmically as well – we show that any game in LNB2, the subclass of 2-person LNB games, is solvable in strongly polynomial time and is rational; however, these algorithms are not combinatorial. Another key class of LNB we define is *submodular utility Nash and nonsymmetric bargaining games*, abbreviated SNB. We show that each game in SNB is rational and is solvable via a combinatorial, strongly polynomial time algorithm. The main conclusion of Chakrabarty et. al. (2009) is that this way of imposing submodularity (a natural economies of scale condition) in games endows them with several desirable properties.