Theorem 1. *Two plus two equals four.*

Proof of Theorem 1. The set of integers is an (infinite) group with respect to addition. Since 2 is an integer, the sum of 2 and 2 must also be an integer.

Suppose, for the sake of contradiction, that $2+2<4$. We have $2>0$. Adding two to both sides, we get $2+2>0+2$. Since 0 is the identity element for addition, we have $2+2>2$. Hence

$$2<2+2<4$$

so $2+2$ must equal 3. Since 3 is prime, by Fermat's Little Theorem we have the following for $a \in \mathbb{Z}^+$:

$$\begin{aligned} a^{3-1} &\equiv 1 \pmod{3} \\ a^{3-1} &\equiv 1 \pmod{2+2} \end{aligned}$$

But

$$a^{3-1}=\frac{a^3}{a}=\frac{a\times a\times a}{a}=a\times a$$

and, for $a=2$,

$$2\times 2\equiv 0 \pmod{2+2}$$

since, by the definition of multiplication, $2\times 2=2+2$. So, we have

$$\begin{aligned} 2^{3-1} &\equiv 0 \pmod{2+2} \\ 2^{2+2-1} &\equiv 0 \pmod{2+2} \end{aligned}$$

This is a contradiction to Fermat's Little Theorem, so $2+2$ must not be prime. But 3 is prime. Hence $2+2$ must not equal 3, and therefore $2+2\geq 4$.

It remains to show that $2+2$ is not greater than 4. To prove this we need the following lemma:

Lemma 1. *$\forall a \in \mathbb{Z}$, if $a>4$, $\exists b \in \mathbb{Z}$ such that $b>0$ and $a-2=2+b$.*

If a were a solution to the equation $2+2=a$, then we would have $a-2=2+0$. The lemma states that this cannot hold for any $a>4$, and so $a=4$, as desired. So, it only remains to prove our lemma.

Proof of Lemma 1. The proof is by induction over a. Our base case is $a = 5$. Let $5-2 = 2+b$. Five is the 5th Fibonacci number, and 2 is the 3rd Fibonacci number. Therefore, by the definition of Fibonacci numbers, $5 - 2$ must be the 4th Fibonacci number. Letting f_i denote the ith Fibonacci number, then we have

$$f_i - f_{i-1} > 0$$

for $i \neq 2$, because $f_2 - f_1 = f_0$ and $f_0 = 0$, but $f_1 = 1$, and the Fibonacci sequence is nondecreasing. Hence $(5 - 2) - 2 > 0$.

Now, suppose that $\exists b \in \mathbb{Z}$ such that $b > 0$ and $(k - 1) - 2 = 2 + b$. We need to prove that, for some $b' > 0$, $k - 2 = 2 + b'$. Our inductive hypothesis is equivalent to:

$$\begin{aligned} k - 1 - 2 &= 2 + b \\ k - 1 - 2 + 1 &= 2 + b + 1 \\ k - 2 &= 2 + (b + 1) \end{aligned}$$

Since $1 > 0$ and $b > 0$, we have $(b + 1) > 0$. Thus, letting $b' = b + 1$, we have a nonnegative solution to $k - 2 = 2 + b'$, as desired. $\square$

By Lemma 1, it is not the case that $2 + 2 > 4$. Hence $2 + 2 \leq 4$. We also have $2 + 2 \geq 4$. Therefore,

$$2 + 2 = 4$$

$\square$