CS202 Final Exam

December 15th, 2004

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are seven problems on this exam, each worth 20 points, for a total of 140 points. You have approximately three hours to complete this exam.

1 A multiplicative game (20 points)

Consider the following game: A player starts with a score of 0. On each turn, the player rolls two dice, each of which is equally likely to come up 1, 2, 3, 4, 5, or 6. They then take the product xy of the two numbers on the dice. If the product is greater than 20, the game ends. Otherwise, they add the product to their score and take a new turn. The player's score at the end of the game is thus the sum of the products of the dice for all turns before the first turn on which they get a product greater than 20.

1. What is the probability that the player's score at the end of the game is zero?

2. What is the expectation of the player's score at the end of the game?

Solution

1. The only way to get a score of zero is to lose on the first roll. There are 36 equally probable outcomes for the first roll, and of these the six outcomes (4,6), (5,5), (5,6), (6,4), (6,5), and (6,6) yield a product greater than 20. So the probability of getting zero is $6/36 = 1/6$.

2. To compute the total expected score, let us first compute the expected score for a single turn. This is

$$\frac{1}{36}\sum_{i=1}^{6}\sum_{j=1}^{6} ij[ij \leq 20].$$

   where $[ij \leq 20]$ is the indicator random variable for the event that $ij \leq 20$.

   I don't know of a really clean way to evaluate the sum, but we can expand it as

$$
\begin{aligned}
& \left(\sum_{i=1}^{3} i\right)\left(\sum_{j=1}^{6} j\right) + 4\sum_{j=1}^{5} j + 5\sum_{j=1}^{4} j + 6\sum_{j=1}^{3} j \\
= \quad & 6 \cdot 21 + 4 \cdot 15 + 5 \cdot 10 + 6 \cdot 6 \\
= \quad & 126 + 60 + 50 + 36 \\
= \quad & 272.
\end{aligned}
$$

So the expected score per turn is $272/36 = 68/9$.

Now we need to calculate the expected total score; call this value S. Assuming we continue after the first turn, the expected total score for the second and subsequent turns is also S, since the structure of the tail of the game is identical to the game as a whole. So we have

$$S = 68/9 + (5/6)S,$$

which we can solve to get $S = (6 \cdot 68)/9 = 136/3$.

2 An equivalence in space (20 points)

Let V be a k-dimensional vector space over the real numbers $\mathbf{R}$ with a standard basis $\vec{x}_i$. Recall that any vector $\vec{z}$ in V can be represented uniquely as $\sum_{i=1}^{k} z_i \vec{x}_i$. Let $f : V \to \mathbf{R}$ be defined by $f(\vec{z}) = \sum_{i=1}^{k} |z_i|$, where the z_i are the coefficients of $\vec{z}$ in the standard representation. Define a relation $\sim$ on $V \times V$ by $\vec{z}_1 \sim \vec{z}_2$ if and only if $f(\vec{z}_1) = f(\vec{z}_2)$. Show that $\sim$ is an equivalence relation, i.e., that it is reflexive, symmetric, and transitive.

Solution

Both the structure of the vector space and the definition of f are irrelevant; the only fact we need is that $\vec{z}_1 \sim \vec{z}_2$ if and only if $f(\vec{z}_1) = f(\vec{z}_2)$. Thus for all $\vec{z}$, $\vec{z} \sim \vec{z}$ since $f(\vec{z}) = f(\vec{z}$ (reflexivity); for all $\vec{y}$ and $\vec{z}$, if $\vec{y} \sim \vec{z}$, then $f(\vec{y}) = f(\vec{z})$ implies $f(\vec{z}) = f(\vec{y})$ implies $\vec{z} \sim \vec{y}$ (symmetry); and for all $\vec{x}$, $\vec{y}$, and $\vec{z}$, if $\vec{x} \sim \vec{y}$ and $\vec{y} \sim \vec{z}$, then $f(\vec{x}) = f(\vec{y})$ and $f(\vec{y}) = f(\vec{z})$, so $f(\vec{x}) = f(\vec{z})$ and $\vec{x} \sim \vec{z}$ (transitivity).

3 A very big fraction (20 points)

Use the fact that $p = 2^{24036583} - 1$ is prime to show that

$$\frac{9^{2^{24036582}} - 9}{2^{24036583} - 1}$$

is an integer.

Solution

Let's save ourselves a lot of writing by letting $x = 24036583$, so that $p = 2^x - 1$ and the fraction becomes

$$\frac{9^{2^{x-1}} - 9}{p}.$$

To show that this is an integer, we need to show that p divides the denominator, i.e., that

$$9^{2^{x-1}} - 9 = 0 \pmod{p}.$$

We'd like to attack this with Fermat's Little Theorem, so we need to get the exponent to look something like $p - 1 = 2^x - 2$. Observe that $9 = 3^2$, so

$$9^{2^{x-1}} = (3^2)^{2^{x-1}} = 3^{2^x} = 3^{2^x-2} \cdot 3^2 = 3^{p-1} \cdot 3^2.$$

But $3^{p-1} = 1 \pmod{p}$, so we get $9^{2^{x-1}} = 3^2 = 9 \pmod{p}$, and thus $9^{2^{x-1}} - 9 = 0 \pmod{p}$ as desired.

4 A pair of odd vertices (20 points)

Let G be a simple undirected graph (i.e., one with no self-loops or parallel edges), and let u be a vertex in G with odd degree. Show that there is another vertex $v \neq u$ in G such that (a) v also has odd degree, and (b) there is a path from u to v in G.

Solution

Let G' be the connected component of u in G. Then G' is itself a graph, and the degree of any vertex is the same in G' as in G. Since the sum of all the degrees of vertices in G' must be even by the Handshaking Lemma, there cannot be an odd number of odd-degree vertices in G', and so there is some v in G' not equal to u that also has odd degree. Since G' is connected, there exists a path from u to v.

5 How many magmas? (20 points)

Recall that a *magma* is an algebra consisting of a set of elements and one binary operation, which is not required to satisfy any constraints whatsoever except closure. Consider a set S of n elements. How many distinct magmas are there that have S as their set of elements?

Solution

Since the carrier is fixed, we have to count the number of different ways of defining the binary operation. Let's call the operation f. For each ordered pair of elements $(x, y) \in S \times S$, we can pick any element $z \in S$ for the value of $f(x, y)$. This gives n choices for each of the n^2 pairs, which gives n^{n^2} magmas on S.

6 A powerful relationship (20 points)

Recall that the powerset $\mathcal{P}(S)$ of a set S is the set of sets $\{A : A \subseteq S\}$. Prove that if $S \subseteq T$, then $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.

Solution

Let $A \in \mathcal{P}(S)$; then by the definition of $\mathcal{P}(S)$ we have $A \subseteq S$. But then $A \subseteq S \subseteq T$ implies $A \subseteq T$, and so $A \in \mathcal{P}(T)$. Since A was arbitrary, $A \in \mathcal{P}(T)$ holds for all A in $\mathcal{P}(S)$, and we have $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.

7 A group of archaeologists (20 points)

Archaeologists working deep in the Upper Nile Valley have discovered a curious machine, consisting of a large box with three levers painted red, yellow, and blue. Atop the box is a display that shows one of set of n hieroglyphs. Each lever can be pushed up or down, and pushing a lever changes the displayed hieroglyph to some other hieroglyph. The archaeologists have determined by extensive experimentation that for each hieroglyph x, pushing the red lever *up* when x is displayed always changes the display to the same hieroglyph $f(x)$, and pushing the red lever *down* always changes hieroglyph $f(x)$ to x. A similar property holds for the yellow and blue levers: pushing yellow up sends x to $g(x)$ and down sends $g(x)$ to x; and pushing blue up sends x to $h(x)$ and down sends $h(x)$ to x.

Prove that there is a finite number k such that no matter which hieroglyph is displayed initially, pushing any one of the levers up k times leaves the display with the same hieroglyph at the end.

Clarification added during exam: $k > 0$.

Solution

Let H be the set of hieroglyphs, and observe that the map $f : H \to H$ corresponding to pushing the red lever up is invertible and thus a permutation. Similarly, the maps g and h corresponding to yellow or blue up-pushes are also permutations, as are the inverses f^-1, g^-1, and h^-1 corresponding to red, yellow, or blue down-pushes. Repeated pushes of one or more levers correspond to compositions of permutations, so the set of all permutations obtained by sequences of zero or more pushes is the subgroup G of the permutation group $S_{|H|}$ generated by f, g, and h.

Now consider the cyclic subgroup $\langle f \rangle$ of G generated by f alone. Since G is finite, there is some index m such that $f^m = e$. Similarly there are indices n and p such that $g^n = e$ and $h^p = e$. So pushing the red lever up any multiple of k times restores the initial state, as does pushing the yellow lever up any multiple of n times or the blue lever up any multiple of p times. Let $k = mnp$. Then k is a multiple of m, n, and p, and pushing any single lever up k times leaves the display in the same state.