Problem Set 2 Solutions

1 Problem 1.6.26

Suppose $A \neq B$ and neither A nor B are empty. We must prove that $A \times B \neq B \times A$. Since $A \neq B$, either we can find an element x that is in A but not B, or vice versa. The two cases are similar, so without loss of generality, let us assume that x is in A but not B. Also, since B is not empty, there is some element $y \in B$. Then (x, y) is in $A \times B$ by definition, but is not in $B \times A$ since $x \notin B$. Therefore $A \times B \neq B \times A$.

2 Problem 1.6.28

a. This is a real number whose cube is -1. This is true, since $x = -1$ is a solution.

b. There is an integer such that the number obtained by adding 1 to it is greater than the integer. This is true – in fact, every integer satisfies this statement.

c. For every integer, the number obtained by subtracting 1 is again an integer. This is true.

d. The square of every integer is an integer. This is true.

3 Problem 1.7.14

a. Suppose that $x \in A \cup B$. Then either $x \in A$ or $x \in B$. In either case, certainly $x \in A \cup B \cup C$.

b. Suppose $x \in A \cap B \cap C$. Then x is in all three of these sets. In particular, it is in both A and B and therefore in $A \cap B$ as desired.

c. Suppose $x \in (A - B) - C$. Then x is in $A - B$ but not in C. Since $x \in A - B$, we know that $x \in A$. Since we have established that $x \in A$ but $x \notin C$, we have proved that $x \in A - C$.

d. To show that the set given on the left-hand side is empty, it suffices to assume that x is some element in that set and derive a contradiction, thereby showing that no x exists. So suppose that $x \in (A - C) \cap (C - B)$. Then $x \in A - C$ and $x \in C - B$. The first of these statements implies that $x \notin C$, while the second implies that $x \in C$. This is impossible, so our proof by contradiction is complete.

e. To establish the equality, we need to prove inclusion in both directions. To prove that $(B-A) \cup (C-A) \subseteq (B \cup C) - A$, suppose that $x \in (B-A) \cup (C-A)$. Then either $x \in (B - A)$ or $x \in (C - A)$. Without loss of generality, assume the former (the proof in the latter case is exactly parallel.) Then $x \in B$ and $x \notin A$. From the first of these assertions, it follows that $x \in B \cup C$. thus we can conclude that $x \in (B \cup C) - A$, as desired. For the converse, that is, to show that $(B \cup C) - A \subseteq (B - A) \cup (C - A)$, suppose that $x \in (B \cup C) - A$. This means that $x \in (B \cup C)$ and $x \notin A$. The first of these assertions tells us that either $x \in B$ or $x \in C$. Thus either $x \in B - A$ or $x \in C - A$. In either case, $x \in (B - A) \cup (C - A)$.

4 Problem 1.8.16

a. $f(n) = n + 17$

b. $f(n) = \lceil n/2 \rceil$.

c. We let $f(n) = n - 1$ for even values of n and $f(n) = n + 1$ for odd values of n. Note this is one function, even though its definition used two formulas, depending on the parity of n.

d. f(n) = 17

5 Problem 1.8.36

a. We need to prove two things. First suppose $x \in f^{-1}(S \cup T)$. This means that $f(x) \in S \cup T$. Therefore either $f(x) \in S$ or $f(x) \in T$. In the first case $x \in f^{-1}(S)$ and in the second case $x \in f^{-1}(T)$. In either case then, $x \in f^{-1}(S) \cup f^{-1}(T)$. Thus we have show that $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$. Conversely, suppose that $x \in f^{-1}(S) \cup f^{-1}(T)$. Then either $x \in f^{-1}(S)$ or $x \in f^{-1}(T)$, so either $f(x) \in S$ or $f(x) \in T$. Thus we know that $f(x) \in S \cup T$, so by definition $x \in f^{-1}(S \cup T)$. This shows that $f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T)$ as desired.

b. This is simlar to part (a). We have $x \in f^{-1}(S \cap T)$ if and only if $f(x) \in S \cap T$, if and only if $f(x) \in S$ and $f(x) \in T$, if and only if $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$, if and only if $x \in f^{-1}(S) \cap f^{-1}(T)$.