Problem Set 2 Solutions

1 Problem 1.6.26
Suppose $A \neq B$ and neither $A$ nor $B$ are empty. We must prove that $A \times B \neq B \times A$. Since $A \neq B$, either we can find an element $x$ that is in $A$ but not $B$, or vice versa. The two cases are similar, so without loss of generality, let us assume that $x$ is in $A$ but not $B$. Also, since $B$ is not empty, there is some element $y \in B$. Then $(x, y)$ is in $A \times B$ by definition, but is not in $B \times A$ since $x \notin B$. Therefore $A \times B \neq B \times A$.

2 Problem 1.6.28
a. This is a real number whose cube is -1. This is true, since $x = -1$ is a solution.

b. There is an integer such that the number obtained by adding 1 to it is greater than the integer. This is true—in fact, every integer satisfies this statement.

c. For every integer, the number obtained by subtracting 1 is again an integer. This is true.

d. The square of every integer is an integer. This is true.

3 Problem 1.7.14
a. Suppose that $x \in A \cup B$. Then either $x \in A$ or $x \in B$. In either case, certainly $x \in A \cup B \cup C$.

b. Suppose $x \in A \cap B \cap C$. Then $x$ is in all three of these sets. In particular, it is in both $A$ and $B$ and therefore in $A \cap B$ as desired.

c. Suppose $x \in (A - B) - C$. Then $x$ is in $A - B$ but not in $C$. Since $x \in A - B$, we know that $x \in A$. Since we have established that $x \in A$ but $x \notin C$, we have proved that $x \in A - C$.

d. To show that the set given on the left-hand side is empty, it suffices to assume that $x$ is some element in that set and derive a contradiction,
thereby showing that no \( x \) exists. So suppose that \( x \in (A - C) \cap (C - B) \). Then \( x \in A - C \) and \( x \in C - B \). The first of these statements implies that \( x \notin C \), while the second implies that \( x \in C \). This is impossible, so our proof by contradiction is complete.

e. To establish the equality, we need to prove inclusion in both directions. To prove that \((B - A) \cup (C - A) \subseteq (B \cup C) - A\), suppose that \( x \in (B - A) \cup (C - A) \). Then either \( x \in (B - A) \) or \( x \in (C - A) \). Without loss of generality, assume the former (the proof in the latter case is exactly parallel.) Then \( x \in B \) and \( x \notin A \). From the first of these assertions, it follows that \( x \in B \cup C \). Thus we can conclude that \( x \in (B \cup C) - A \), as desired. For the converse, that is, to show that \((B \cup C) - A \subseteq (B - A) \cup (C - A)\), suppose that \( x \in (B \cup C) - A \). This means that \( x \in (B \cup C) \) and \( x \notin A \). The first of these assertions tells us that either \( x \in B \) or \( x \in C \). Thus either \( x \in B - A \) or \( x \in C - A \). In either case, \( x \in (B - A) \cup (C - A) \).

4 Problem 1.8.16

a. \( f(n) = n + 17 \)

b. \( f(n) = \lceil n/2 \rceil \).

c. We let \( f(n) = n - 1 \) for even values of \( n \) and \( f(n) = n + 1 \) for odd values of \( n \). Note this is one function, even though its definition used two formulas, depending on the parity of \( n \).

d. \( f(n) = 17 \)

5 Problem 1.8.36

a. We need to prove two things. First suppose \( x \in f^{-1}(S \cup T) \). This means that \( f(x) \in S \cup T \). Therefore either \( f(x) \in S \) or \( f(x) \in T \). In the first case \( x \in f^{-1}(S) \) and in the second case \( x \in f^{-1}(T) \). In either case then, \( x \in f^{-1}(S) \cup f^{-1}(T) \). Thus we have show that \( f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T) \). Conversely, suppose that \( x \in f^{-1}(S) \cup f^{-1}(T) \). Then either \( x \in f^{-1}(S) \) or \( x \in f^{-1}(T) \), so either \( f(x) \in S \) or \( f(x) \in T \). Thus we know that \( f(x) \in S \cup T \), so by definition \( x \in f^{-1}(S \cup T) \). This shows that \( f^{-1}(S) \cup f^{-1}(T) \subseteq f^{-1}(S \cup T) \) as desired.

b. This is similar to part (a). We have \( x \in f^{-1}(S \cap T) \) if and only if \( f(x) \in S \cap T \) if and only if \( f(x) \in S \) and \( f(x) \in T \), if and only if \( x \in f^{-1}(S) \) and \( x \in f^{-1}(T) \), if and only if \( x \in f^{-1}(S) \cap f^{-1}(T) \).