Problem Set 3 Solutions

1 Problem 3.2.8

One rule would be that each term is 2 greter than the previous term; this would generate the sequence $3, 5, 7, 9, \ldots$. Another rule would be that the nth term is the nth odd prime; this would generate the sequence $3, 5, 7, 11, 13, \ldots$. Another rule would be the ith term is the same as the $i \mod 3$th term. This would generate the sequence $3, 5, 7, 3, 5, 7, \ldots$.

2 Problem 3.2.10

a. The first term is 3, and the nth term is obtained by adding $2n - 1$ to the previous term. In other words, we successively add 3, then 5, then 7, and so on. Alternatively, we see that the nth term is $n^2 + 2$; we can see this by inspection if we happen to notice how close each term is to a perfect square, or we can fit a quadratic polynomial to the data.

b. This is an arithmetic sequence whose first term is 7 and whose difference is 4. Thus, the nth term is $7 + 4(n - 1) = 4n + 3$.

c. The nth term is clearly the binary expansion of n.

d. The sequence consists of one 1, followed by three 3s, followed by five 5s, and so on, with the rule that the first two values are 1 and 3 and each subsequent value is the sum of the previous two values. obviously other answers are possible as well.

e. If we stare at this equence long enough, we notice that the nth term is $3^n - 1$.

f. We notice that each term evenly divies the next, and the multipliers are successively 3, 5, 7, 9, 11, and so on.

g. The sequence consists of one 1, followed by two 0s, then three 1s, four 0s, five 1s, and so on, alternating between 0s and 1s and having one more item in each group than in the previous group.

h. Each term is the square of its predecessor.

3 Problem 3.2.18

We will just write out the sums explicitly in each case:

a. $(1-1)+(1-2)+(2-1)+(2-2)+(3-1)+(3-2)=3$

b. $(0+0)+(0+2)+(0+4)+(3+0)+(3+2)+(3+4)+(6+0)+(6+2)+(6+4)+(9+0)+(9+2)+(9+4)=78$

c. $(0+1+2)+(0+1+2)+(0+1+2)=9$

d. $(0+0+0+0)+(0+1+8+27)+(0+4+32+108)=180$

4 Problem 3.3.36

We proceed by induction. The base case is true, since $1 \cdot 2^1 = (1-1)2^{1+1}+2$. We assume the inductive hypothesis, that

$$\sum_{k=1}^{n} k \cdot 2^k = (n-1)2^{n+1}+2,$$

and try to prove that

$$\sum_{k=1}^{n+1} k \cdot 2^k = n2^{n+2}+2.$$

Splitting the left-hand side into its first n terms followed by its last term and invoking the inductive hypothesis, we have

$$\begin{aligned}
\sum_{k=1}^{n+1} k \cdot 2^k &= \left(\sum_{k=1}^{n} k \cdot 2^k\right) + (n+1)2^{n+1} \\
&= (n-1)2^{n+1}+2+(n+1)2^{n+1} \\
&= 2n \cdot 2^{n+1}+2 \\
&= n \cdot 2^{n+2}+2
\end{aligned}$$

as desired.

5 Problem 3.3.46

These are really easier to do directly than by induction.

a. Suppose that $x \in \cup_{k=1}^{n} A_k$. Then $x \in A_k$ for some k. Since $A_k \subseteq B_k$, we know that $x \in B_k$., Therefore by definition $x \in \cup_{k=1}^{n} B_k$ as desired.

b. Suppose that $x \in \cap_{k=1}^{n}$. Then $x \in A_k$ for all k. Since $A_k \subseteq B_k$, we know that $x \in B_k$ for all k. Therefore by definition, $x \in \cap_{k=1}^{n} B_k$ as desired.