Problem Set 9 Solutions

1 Maximization

1.1 Part a

In order to prove that any subset $S \subseteq \mathbf{N}$ yields a subsemigroup of A, we need to prove that S is closed under the operation max. This follows from the fact that if $a, b \in S$, then $\max(a, b) = a$ or b. Thus, $\max(a, b) \in S$.

1.2 Part b

The subsets of $\mathbf{N}$ that yield submonoids are all subsets that contain the identity element, 0.

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2.1 Part a

We provide a counterexample. Let A be the egalitarian semigroup with carrier $\{a, b\}$ and operation defined by $xy = a$ for all $x, y \in A$.

We define the function $f : A \to A$ by $f(a) = b$, $f(b) = a$. Then $f(ab) = f(a) = b$ but $f(a)f(b) = ab = a$. Thus, f is not a homomorphism.

2.2 Part b

Let $F(S)$ be carried by $S \cup \{x\}$, where $x \notin S$, and define $ab = x$ for all $a, b \in F(S)$. We claim that F defines the free algebra for egalitarian semigroups.

Thus, suppose G is any egalitarian semigroup and let $y \in G$ be the element such that $ab = y$ for all $a, b \in G$. Given $f : S \to G$, define f^* by $f^*(a) = f(a)$ if $a \in S$, and $f^*(x) = y$. Then $f^*(ab) = f^*(x) = y = f^*(a)f^*(b)$ for all $a, b \in S$. Thus, f^* is a homomorphism. In order to show that f^* is unique, let $g^* : F(S) \to G$ be a homomorphism such that $g^*(a) = f(a)$ for all $a \in S$. Then note that $g^*(x) = g^*(ab) = g^*(a)g^*(b) = y = f^*(x)$. Thus, $f^*(z) = g^*(z)$ for all $z \in F(S)$.

3 Quotient

3.1 Part a

If $x \in A$ is a string that contains k b's, then note that $f(x) = b^k$. Thus, if x, y are two strings in A with n_1 and n_2 b's respectively, then the string xy contains $n_1 + n_2$ b's. $f(x)f(y) = b^{n_1+n_2} = f(x + y)$.

3.2 Part b

By applying the first isomorphism theorem (Theorem 5 from the notes), we see that $A/\ker(f)$ is isomorphic to $f(A) = B$. Thus, it suffices to prove that B is isomorphic to $(N, +, 0)$. We claim that the function $h : B \to N$ defined by $h(x) = length(x)$ is a bijective homomorphism. Let x, y be two strings in B with n_1 and n_2 b's, respectively. Note that $x \circ y$ is a string of length $n_1 + n_2$. Then $h(x) + h(y) = n_1 + n_2 = h(x \circ y)$. It's easy to see that h is bijective, and that h^{-1} is a homomorphism.

4 Back to the Center

In order to prove that C is a subgroup, we must prove that it is closed, contains the identity, and contains the inverse of each element. Associativity we get for free, because G is a group.

1. Claim: C is closed.

   Proof: Suppose $x, y \in C$ and $a \in G$. Then, $xya = xay = axy$. Thus, $xy \in C$.

2. Claim: C contains the identity.

   Proof: Let $x \in G$. Note that $ex = x = xe$.

3. Claim: C contains inverses.

   Proof: Suppose $x \in C$ and $a \in G$. Then $x^{-1}a = x^{-1}axx^{-1} = x^{-1}xax^{-1} = ax^{-1}$.

By the definition of C, C is clearly commutative. Therefore C is an abelian subgroup of G.