# CS202 Final Exam

December 16th, 2005

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are six problems on this exam, each worth 20 points, for a total of 120 points. You have approximately three hours to complete this exam.

# 1 Order (20 points)

Recall that the *order* of an element $x$ of a group is the least positive integer $k$ such that $x^k = e$, where $e$ is the identity, or $\infty$ if no such $k$ exists.

Prove or disprove: In the symmetric group $S_n$ of permutations on $n$ elements, the order of any permutation is at most $\binom{n}{2}$.

### Clarifications added during exam

- Assume $n > 2$.

### Solution

Disproof: Consider the permutation $(1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11\ 12\ 13\ 14\ 15\ 16\ 17)$ in $S_{17}$. This has order $2 \cdot 3 \cdot 5 \cdot 7 = 210$ but $\binom{17}{2} = \frac{17 \cdot 16}{2} = 136$.

# 2 Count the subgroups (20 points)

Recall that the *free group* over a singleton set $\{a\}$ consists of all words of the form $a^k$, where $k$ is an integer, with multiplication defined by $a^k a^m = a^{k+m}$.

Prove or disprove: The free group over $\{a\}$ has exactly one finite subgroup.

### Solution

Proof: Let $F$ be the free group defined above and let $S$ be a subgroup of $F$. Suppose $S$ contains $a^k$ for some $k \neq 0$. Then $S$ contains $a^{2k}, a^{3k}, \ldots$ because it is closed under multiplication. Since these elements are all distinct, $S$ is infinite.

The alternative is that $S$ does not contain $a^k$ for any $k \neq 0$; this leaves only $a^0$ as possible element of $S$, and there is only one such subgroup: the trivial subgroup $\{a^0\}$.

# 3 Two exits (20 points)

Let $G = (V, E)$ be a nonempty connected undirected graph with no self-loops or parallel edges, in which every vertex has degree 4. Prove or disprove: For any partition of the vertices $V$ into two nonempty non-overlapping subsets $S$ and $T$, there are at least two edges that have one endpoint in $S$ and one in $T$.

### Solution

Proof: Because $G$ is connected and every vertex has even degree, there is an Euler tour of the graph (a cycle that uses every edge exactly once). Fix some particular tour and consider a partition of $V$ into two sets $S$ and $T$. There must be at least one edge between $S$ and $T$, or $G$ is not connected; but if there is only one, then the tour can't return to $S$ or $T$ once it leaves. It follows that there are at least 2 edges between $S$ and $T$ as claimed.

## 4 Victory (20 points)

A sabermetrician wishes to test the hypothesis that a set of $n$ baseball teams are stricty ranked, so that no two teams have the same rank and if some team $A$ has a higher rank than some team $B$, $A$ will always beat $B$ in a 7-game series. To test this hypothesis, the sabermetrician has each team play a 7-game series against each other team.

Suppose that the teams are in fact all equally incompetent and that the winner of each series is chosen by an independent fair coin-flip. What is the probability that the results will nonetheless be consistent with some strict ranking?

### Solution

Each ranking is a total order on the $n$ teams, and we can describe such a ranking by giving one of the $n!$ permutations of the teams. These in turn generate $n!$ distinct outcomes of the experiment that will cause the sabermetrician to believe the hypothesis. To compute the probability that one of these outcomes occurs, we must divide by the total number of outcomes, giving

$$\Pr\left[\text{strict ranking}\right] = \frac{n!}{2^{\binom{n}{2}}}.$$

## 5 An aggressive aquarium (20 points)

A large number of juvenile piranha, weighing 1 unit each, are placed in an aquarium. Each day, each piranha attempts to eat one other piranha. If successful, the eater increases its weight to the sum of its previous weight and the weight of its meal (and the eaten piranha is gone); if unsuccessful, the piranha remains at the same weight.

Prove that after $k$ days, no surviving piranha weighs more than $2^k$ units.

### Clarifications added during exam

- It is not possible for a piranha to eat and be eaten on the same day.

### Solution

By induction on $k$. The base case is $k = 0$, when all piranha weigh exactly $2^0 = 1$ unit. Suppose some piranha has weight $x \leq 2^k$ after $k$ days. Then either its weight stays the same, or it successfully eats another piranha of weight $y \leq 2^k$ increases its weight to $x + y \leq 2^k + 2^k = 2^{k+1}$. In either case the claim follows for $k + 1$.

## 6 A subspace of matrices (20 points)

Recall that a subspace of a vector space is a set that is closed under vector addition and scalar multiplication. Recall further that the subspace *generated* by a set of vector space elements is the smallest such subspace, and its *dimension* is the size of any basis of the subspace.

Let $A$ be the 2-by-2 matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

over the reals, and consider the subspace $S$ of the vector space of 2-by-2 real matrices generated by the set $\{A, A^2, A^3, \ldots\}$. What is the dimension of $S$?

### Solution

First let's see what $A^k$ looks like. We have

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

and in general we can show by induction that

$$A^k = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & k-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Observe now that for any $k$,

$$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = (k-1) \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} - (k-2) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (k-1)A^2 - (k-2)A.$$

It follows that $\{A, A^2\}$ generates all the $A^k$ and thus generates any linear combination of the $A^k$ as well. It is easy to see that $A$ and $A^2$ are linearly independent: if $c_1 A + c_2 A^2 = 0$, we must have (a) $c_1 + c_2 = 0$ (to cancel out the diagonal entries) and (b) $c_1 + 2c_2 = 0$ (to cancel out the nonzero off-diagonal entry). The only solution to both equations is $c_1 = c_2 = 0$.

Because $\{A, A^2\}$ is a linearly independent set that generates $S$, it is a basis, and $S$ has dimension 2.