CS202 Midterm Exam Solutions

October 12th, 2005

Write your answers on the exam. Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately 50 minutes to complete this exam.

1 A recurrence (20 points)

Give a simple formula for $T(n)$, where:

$$\begin{aligned} T(0) &= 1. \\ T(n) &= 3T(n-1) + 2^n, \text{when } n > 0. \end{aligned}$$

Solution

Using generating functions

Let $F(z) = \sum_{n=0}^{\infty} T(n) z^n$, then

$$F(z) = 3zF(z) + \frac{1}{1-2z}.$$

Solving for F gives

$$\begin{aligned} F(z) &= \frac{1}{(1-2z)(1-3z)} \\ &= \frac{-2}{1-2z} + \frac{3}{1-3z}. \end{aligned}$$

From the generating function we can immediately read off

$$T(n) = 3 \cdot 3^n - 2 \cdot 2^n = 3^{n+1} - 2^{n+1}.$$

Without using generating functions

It is possible to solve this problem without generating functions, but it's harder. Here's one approach based on forward induction. Start by computing the first few values of $T(n)$. We'll avoid reducing the expressions to make it easier to spot a pattern.

$$\begin{aligned}
T(0) &= 1 \\
T(1) &= 3+2 \\
T(2) &= 3^2+3\cdot 2+2^2 \\
T(3) &= 3^3+3^2\cdot 2+3\cdot 2^2+2^3 \\
T(4) &= 3^4+3^3\cdot 2+3^2\cdot 2^2+3\cdot 2^3+2^4
\end{aligned}$$

At this point we might guess that

$$T(n) = \sum_{k=0}^{n} 3^{n-k}2^k = 3^n \sum_{k=0}^{n} (2/3)^k = 3^n \left(\frac{1-(2/3)^{n+1}}{1-(2/3)} \right) = 3^{n+1} - 2^{n+1}.$$

A guess is not a proof; to prove that this guess works we verify $T(0) = 3^1 - 2^1 = 3 - 2 = 1$ and $T(n) = 3T(n-1) + 2^n = 3(3^n - 2^n) + 2^n = 3^{n+1} - 2 \cdot 2^n = 3^{n+1} - 2^{n+1}$.

2 An induction proof (20 points)

Prove by induction on n that $n! > 2^n$ for all integers $n \geq n_0$, where n_0 is an integer chosen to be as small as possible.

Solution

Trying small values of n gives $0! = 1 = 2^0$ (bad), $1! = 1 < 2^1$ (bad), $2! = 2 < 2^2$ (bad), $3! = 6 < 2^3$ (bad), $4! = 24 > 2^4 = 16$ (good). So we'll guess $n_0 = 4$ and use the $n = 4$ case as a basis.

For larger n, we have $n! = n(n-1) > n2^{n-1} > 2 \cdot 2^{n-1} = 2^n$.

3 Some binomial coefficients (20 points)

Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$ when $1 \leq k \leq n$.

Solution

There are several ways to do this. The algebraic version is probably cleanest.

Combinatorial version

The LHS counts the way to choose k of n elements and then specially mark one of the k. Alternatively, we could choose the marked element first (n choices) and then choose the remaining $k-1$ elements from the remaining $n-1$ elements ($\binom{n-1}{k-1}$ choices); this gives the RHS.

Algebraic version

Compute $k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = n\binom{n-1}{k-1}$.

Generating function version

Observe that $\sum_{k=0}^{n} k\binom{n}{k} z^k = z\frac{d}{dz}(1+z)^n = zn(1+z)^{n-1} = \sum_{k=0}^{n-1} n\binom{n-1}{k} z^{k+1} = \sum_{k=1}^{n} n\binom{n-1}{k-1} z^k$. Now match z^k coefficients to get the desired result.

4 A probability problem (20 points)

Suppose you flip a fair coin n times, where $n \geq 1$. What is the probability of the event that both of the following hold: (a) the coin comes up heads at least once and (b) once it comes up heads, it never comes up tails on any later flip?

Solution

Let A_i be the event that the coin comes up heads for the first time on flip i, for each $i \in \{1 \ldots n\}$. Then the desired event is the disjoint union of the A_i. Since each A_i is a single sequence of coin-flips, each occurs with probability 2^{-n}. Summing over all i gives a total probability of $n2^{-n}$.