CS202 Final Exam

December 19th, 2008

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are five problems on this exam, each worth 20 points, for a total of 100 points. You have approximately three hours to complete this exam.

1 Some logical sets (20 points)

Let A, B, and C be sets.

Prove or disprove: If, for all x, $x \in A \rightarrow (x \in B \rightarrow x \in C)$, then $A \cap B \subseteq C$.

Solution

Proof: Rewrite $x \in A \rightarrow (x \in B \rightarrow x \in C)$ as $x \notin A \vee (x \notin B \vee x \in C)$ or $(x \notin A \vee x \notin B) \vee x \in C$. Applying De Morgan's law we can convert the first OR into an AND to get $\neg(x \in A \wedge x \in B) \vee x \in C$. This can further be rewritten as $(x \in A \wedge x \in B) \rightarrow x \in C$.

Now suppose that this expression is true for all x and consider some x in $A \cap B$. Then $x \in A \wedge x \in B$ is true. It follows that $x \in C$ is also true. Since this holds for every element x of $A \cap B$, we have $A \cap B \subseteq C$.

2 Modularity (20 points)

Let m be an integer greater than or equal to 2. For each a in $\mathbb{Z}_m$, let $f_a : \mathbb{Z}_m \rightarrow \mathbb{Z}_m$ be the function defined by the rule $f_a(x) = ax$.

Show that f_a is a bijection if and only if $\gcd(a, m) = 1$.

Solution

From the extended Euclidean algorithm we have that if $\gcd(a, m) = 1$, then there exists a multiplicative inverse a^{-1} such that $a^{-1}ax = x \pmod{m}$ for all x in $\mathbb{Z}_m$. It follows that f_a has an inverse function $f_{a^{-1}}$, and is thus a bijection.

Alternatively, suppose $\gcd(a, m) = g \neq 1$. Then $f_a(m/g) = am/g = m(a/g) = 0 = a \cdot 0 = f_a(0) \pmod{m}$ but $m/g \neq 0 \pmod{m}$ since $0 < m/g < m$. It follows that f_a is not injective and thus not a bijection.

3 Coin flipping (20 points)

Take a biased coin that comes up heads with probability p and flip it $2n$ times.

What is the probability that at some time during this experiment two consecutive coin-flips come up both heads or both tails?

Solution

It's easier to calculate the probability of the event that we never get two consecutive heads or tails, since in this case there are only two possible patterns of coin-flips: $HTHT\ldots$ or $THTH\ldots$. Since each of these patterns contains exactly n heads and n tails, they occur with probability $p^n(1-p)^n$, giving a total probability of $2p^n(1-p)^n$. The probability that neither sequence occurs is then $1-2p^n(1-p)^n$.

4 A transitive graph (20 points)

Let G be a graph with n vertices on which the adjacency relation is transitive: whenever there is an edge uv and an edge vw, there is also an edge uw. Suppose further that G is connected. How many edges does G have?

Solution

The graph G has exactly $\binom{n}{2}$ edges. The reason is that under the stated conditions, G is a complete graph.

Consider any two vertices u and v. Because G is connected, there is a path $u = v_0v_1\ldots v_k = v$ starting at u and ending at v. We can easily prove by induction that there is an edge uv_i for each $1 \le i \le k$. The existence of the first such edge is immediate from its presence in the path. For later edges, we have from the induction hypothesis that there is an edge uv_i, from the path that there is an edge v_iv_{i+1}, and thus from the transitivity condition that there is and edge uv_{i+1}. When $i = k$, we have that there is an edge uv.

5 A possible matrix identity (20 points)

Prove or disprove: If A and B are symmetric matrices of the same dimension, then $A^2 - B^2 = (A-B)(A+B)$.

Solution

Observe first that $(A - B)(A + B) = A^2 + AB - BA + B^2$. The question then is whether $AB = BA$. Because A and B are symmetric, we have that $BA = B'A' = (AB)'$. So if we can show that AB is also symmetric, then we have $AB = (AB)' = BA$. Alternatively, if we can find symmetric matrices A and B such that AB is *not* symmetric, then $A^2 - B^2 \neq (A - B)(A + B)$.

Let's try multiplying two generic symmetric 2-by-2 matrices:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} d & e \\ e & f \end{pmatrix} = \begin{pmatrix} ad + be & ae + bf \\ bd + ce & be + cf \end{pmatrix}$$

The product doesn't look very symmetric, and in fact we can assign variables to make it not so. We need $ae + bf \neq bd + ce$. Let's set $b = 0$ to make the bf and bd terms drop out, and $e = 1$ to leave just a and c. Setting $a = 0$ and $c = 1$ then gives an asymmetric product. Note that we didn't determine d or f, so let's just set them to zero as well to make things as simple as possible. The result is:

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Which is clearly not symmetric. So for these particular matrices we have $A^2 - B^2 \neq (A - B)(A + B)$, disproving the claim.