1 Some logical sets (20 points)

Let $A$, $B$, and $C$ be sets.

Prove or disprove: If, for all $x$, $x \in A \rightarrow (x \in B \rightarrow x \in C)$, then $A \cap B \subseteq C$.

2 Modularity (20 points)

Let $m$ be an integer greater than or equal to 2. For each $a$ in $\mathbb{Z}_m$, let $f_a : \mathbb{Z}_m \rightarrow \mathbb{Z}_m$ be the function defined by the rule $f_a(x) = ax$.

Show that $f_a$ is a bijection if and only if gcd$(a, m) = 1$.

3 Coin flipping (20 points)

Take a biased coin that comes up heads with probability $p$ and flip it $2n$ times.

What is the probability that at some time during this experiment two consecutive coin-flips come up both heads or both tails?

4 A transitive graph (20 points)

Let $G$ be a graph with $n$ vertices on which the adjacency relation is transitive: whenever there is an edge $uv$ and an edge $vw$, there is also an edge $uw$. Suppose further that $G$ is connected. How many edges does $G$ have?

5 A possible matrix identity (20 points)

Prove or disprove: If $A$ and $B$ are symmetric matrices of the same dimension, then $A^2 - B^2 = (A - B)(A + B)$.