CS202 Midterm Exam

October 24th, 2008

Write your answers on the exam. Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately 50 minutes to complete this exam.

1 Some sums (20 points)

Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be sequences such that for i in $\mathbb{N}$, $a_i \leq b_i$.

Let $A_i = \sum_{j=0}^{i} a_j$ and let $B_i = \sum_{j=0}^{i} b_j$.

Prove or disprove: For all i in $\mathbb{N}$, $A_i \leq B_i$.

Solution

Proof: By induction on i. For $i = 0$ we have $A_0 = a_0 \leq b_0 = B_0$. Now suppose $A_i \leq B_i$. Then $A_{i+1} = \sum_{j=0}^{i+1} a_j = \sum_{j=0}^{i} a_j + a_{i+1} = A_i + a_{i+1} \leq B_i + b_{i+1} = \sum_{j=0}^{i} b_j + b_{j+1} = \sum_{j=0}^{i+1} b_j = B_j$.

2 Nested ranks (20 points)

You are recruiting people for a secret organization, from a population of n possible recruits. Out of these n possible recruits, some subset M will be members. Out of this subset M, some further subset C will be members of the inner circle. Out of this subset C, some further subset X will be Exalted Grand High Maharajaraja Panjandrums of Indifference. It is possible that any or all of these sets will be empty.

If the roster of your organization gives the members of the sets M, C, and X, and if (as usual) order doesn't matter within the sets, how many different possible rosters can you have?

Solution

There is an easy way to solve this, and a hard way to solve this.

Easy way: For each possible recruit x, we can assign x one of four states: non-member, member but not inner circle member, inner circle member but not EGHMPoI, or EGHMPoI. If we know the state of each possible recruit, that determines the contents of M, C, X and vice versa. It follows that there is a one-to-one mapping between these two representations, and that the number of rosters is equal to the number of assignments of states to all n potential recruits, which is 4^n.

Hard way: By repeated application of the binomial theorem. Expressing the selection process in terms of choosing nested subsets of m, c, and x members, the number of possible rosters is

$$\begin{aligned}
\sum_{m=0}^{n} \left\{ \binom{n}{m} \sum_{c=0}^{m} \left[\binom{m}{c} \sum_{x=0}^{c} \binom{c}{x} \right] \right\} &= \sum_{m=0}^{n} \left\{ \binom{n}{m} \sum_{c=0}^{m} \binom{m}{c} 2^c \right\} \\
&= \sum_{m=0}^{n} \binom{n}{m} (1+2)^m \\
&= \sum_{m=0}^{n} \binom{n}{m} 3^m \\
&= (1+3)^n \\
&= 4^n.
\end{aligned}$$

3 Nested sets (20 points)

Let A, B, and C be sets.

1. Prove or disprove: If $A \in B$, and $B \subseteq C$, then $A \subseteq C$.
2. Prove or disprove: If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.

Solution

1. Disproof: Let $A = \{\emptyset\}$, $B = \{A\} = \{\{\emptyset\}\}$, and $C = B$. Then $A \in B$ and $B \subseteq C$, but $A \not\subseteq C$, because $\emptyset \in A$ but $\emptyset \notin C$.
2. Proof: Let $x \in A$. Then since $A \subseteq B$, we have $x \in B$, and since $B \subseteq C$, we have $x \in C$. It follows that every x in A is also in C, and that A is a subset of C.

4 An efficient grading method (20 points)

A test is graded on a scale of 0 to 80 points. Because the grading is completely random, your grade can be represented by a random variable X with $0 \le X \le 80$ and $\mathrm{E}[X] = 60$.

1. What is the maximum possible probability that $X = 80$?

2. Suppose that we change the bounds to $20 \leq X \leq 80$, but $\mathrm{E}[X]$ is still 60. Now what is the maximum possible probability that $X = 80$?

Solution

1. Here we apply Markov's inequality: since $X \geq 0$, we have $\Pr[X \geq 80] \leq \frac{\mathrm{E}[X]}{80} = \frac{60}{80} = 3/4$. This maximum is achieved exactly by letting $X = 0$ with probability $1/4$ and 80 with probability $3/4$, giving $\mathrm{E}[X] = (1/4) \cdot 0 + (3/4) \cdot 80 = 60$.

2. Raising the minimum grade to 20 knocks out the possibility of getting 0, so our previous distribution doesn't work. In this new case we can apply Markov's inequality to $Y = X - 20 \geq 0$, to get $\Pr[X \geq 80] = \Pr[Y \geq 60] \leq \frac{\mathrm{E}[Y]}{60} = \frac{40}{60} = 2/3$. So the extreme case would seem to be that we get 20 with probability $1/3$ and 80 with probability $2/3$. It's easy to check that we then get $\mathrm{E}[X] = (1/3) \cdot 20 + (2/3) \cdot 80 = 180/3 = 60$. So in fact the best we can do now is a probability of $2/3$ of getting 80, less than we had before.