CS202 Final Exam

December 15th, 2004

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are seven problems on this exam, each worth 20 points, for a total of 140 points. You have approximately three hours to complete this exam.

1 A multiplicative game (20 points)

Consider the following game: A player starts with a score of 0. On each turn, the player rolls two dice, each of which is equally likely to come up 1, 2, 3, 4, 5, or 6. They then take the product xy of the two numbers on the dice. If the product is greater than 20, the game ends. Otherwise, they add the product to their score and take a new turn. The player's score at the end of the game is thus the sum of the products of the dice for all turns before the first turn on which they get a product greater than 20.

1. What is the probability that the player's score at the end of the game is zero?
2. What is the expectation of the player's score at the end of the game?

2 An equivalence in space (20 points)

Let V be a k-dimensional vector space over the real numbers $\mathbf{R}$ with a standard basis $\vec{x}_i$. Recall that any vector $\vec{z}$ in V can be represented uniquely as $\sum_{i=1}^{k} z_i \vec{x}_i$. Let $f : V \to \mathbf{R}$ be defined by $f(\vec{z}) = \sum_{i=1}^{k} |z_i|$, where the z_i are the coefficients of $\vec{z}$ in the standard representation. Define a relation $\sim$ on $V \times V$ by $\vec{z}_1 \sim \vec{z}_2$ if and only if $f(\vec{z}_1) = f(\vec{z}_2)$. Show that $\sim$ is an equivalence relation, i.e., that it is reflexive, symmetric, and transitive.

3 A very big fraction (20 points)

Use the fact that $p = 2^{24036583} - 1$ is prime to show that

$$\frac{9^{2^{24036582}} - 9}{2^{24036583} - 1}$$

is an integer.

4 A pair of odd vertices (20 points)

Let G be a simple undirected graph (i.e., one with no self-loops or parallel edges), and let u be a vertex in G with odd degree. Show that there is another vertex $v \neq u$ in G such that (a) v also has odd degree, and (b) there is a path from u to v in G.

5 How many magmas? (20 points)

Recall that a *magma* is an algebra consisting of a set of elements and one binary operation, which is not required to satisfy any constraints whatsoever except closure. Consider a set S of n elements. How many distinct magmas are there that have S as their set of elements?

6 A powerful relationship (20 points)

Recall that the powerset $\mathcal{P}(S)$ of a set S is the set of sets $\{A : A \subseteq S\}$. Prove that if $S \subseteq T$, then $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.

7 A group of archaeologists (20 points)

Archaeologists working deep in the Upper Nile Valley have discovered a curious machine, consisting of a large box with three levers painted red, yellow, and blue. Atop the box is a display that shows one of set of n hieroglyphs. Each lever can be pushed up or down, and pushing a lever changes the displayed hieroglyph to some other hieroglyph. The archaeologists have determined by extensive experimentation that for each hieroglyph x, pushing the red lever *up* when x is displayed always changes the display to the same hieroglyph $f(x)$, and pushing the red lever *down* always changes hieroglyph $f(x)$ to x. A similar property holds for the yellow and blue levers: pushing yellow up sends x to $g(x)$ and down sends $g(x)$ to x; and pushing blue up sends x to $h(x)$ and down sends $h(x)$ to x.

Prove that there is a finite number k such that no matter which hieroglyph is displayed initially, pushing any one of the levers up k times leaves the display with the same hieroglyph at the end.

Clarification added during exam: $k > 0$.