CS202 Final Exam

December 16th, 2005

Write your answers in the blue book(s). Justify your answers. Work alone. Do not use any notes or books.

There are six problems on this exam, each worth 20 points, for a total of 120 points. You have approximately three hours to complete this exam.

1 Order (20 points)

Recall that the *order* of an element x of a group is the least positive integer k such that $x^k = e$, where e is the identity, or ∞ if no such k exists.

Prove or disprove: In the symmetric group S_n of permutations on n elements, the order of any permutation is at most $\binom{n}{2}$.

Clarifications added during exam

- Assume $n > 2$.

2 Count the subgroups (20 points)

Recall that the *free group* over a singleton set $\{a\}$ consists of all words of the form a^k, where k is an integer, with multiplication defined by $a^k a^m = a^{k+m}$.

Prove or disprove: The free group over $\{a\}$ has exactly one finite subgroup.

3 Two exits (20 points)

Let $G = (V, E)$ be a nonempty connected undirected graph with no self-loops or parallel edges, in which every vertex has degree 4. Prove or disprove: For any partition of the vertices V into two nonempty non-overlapping subsets S and T, there are at least two edges that have one endpoint in S and one in T.

4 Victory (20 points)

A sabermetrician wishes to test the hypothesis that a set of n baseball teams are stricty ranked, so that no two teams have the same rank and if some team A has a higher rank than some team B, A will always beat B in a 7-game series. To test this hypothesis, the sabermetrician has each team play a 7-game series against each other team.

Suppose that the teams are in fact all equally incompetent and that the winner of each series is chosen by an independent fair coin-flip. What is the probability that the results will nonetheless be consistent with some strict ranking?

5   An aggressive aquarium (20 points)

A large number of juvenile piranha, weighing 1 unit each, are placed in an aquarium. Each day, each piranha attempts to eat one other piranha. If successful, the eater increases its weight to the sum of its previous weight and the weight of its meal (and the eaten piranha is gone); if unsuccessful, the piranha remains at the same weight.

Prove that after k days, no surviving piranha weighs more than 2^k units.

Clarifications added during exam

- It is not possible for a piranha to eat and be eaten on the same day.

6   A subspace of matrices (20 points)

Recall that a subspace of a vector space is a set that is closed under vector addition and scalar multiplication. Recall further that the subspace *generated* by a set of vector space elements is the smallest such subspace, and its *dimension* is the size of any basis of the subspace.

Let A be the 2-by-2 matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

over the reals, and consider the subspace S of the vector space of 2-by-2 real matrices generated by the set $\{A, A^2, A^3, \ldots\}$. What is the dimension of S?