CS202 Final Exam

December 14th, 2010

Write your answers in the blue book(s). Justify your answers. Give closed-form solutions when possible. Work alone. Do not use any notes or books.

There are five problems on this exam, each worth 20 points, for a total of 100 points. You have approximately three hours to complete this exam.

1 Backwards and forwards (20 points)

Let $\{0,1\}^n$ be the set of all binary strings $x_1 x_2 \dots x_n$ of length n.

For any string x in $\{0,1\}^n$, let $r(x) = x_n x_{n-1} \dots x_1$ be the reversal of x. Let $x \sim y$ if $x = y$ or $x = r(y)$.

Given a string x in $\{0,1\}^n$ and a permutation π of $\{1,\dots,n\}$, let $\pi(x)$ be the string $x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)}$. Let $x \approx y$ if there exists some π such that $x = \pi(y)$.

Both $\sim$ and $\approx$ are equivalence relations. Let $\{0,1\}^n/\sim$ and $\{0,1\}^n/\approx$ be the corresponding sets of equivalence classes.

1. What is $|\{0,1\}^n/\sim|$ as a function of n?
2. What is $|\{0,1\}^n/\approx|$ as a function of n?

Solution

1. Given a string x, the equivalent class $[x] = \{x, r(x)\}$ has either one element (if $x = r(x)$) or two elements (if $x \neq r(x)$). Let m_1 be the number of one-element classes and m_2 the number of two-element classes. Then $|\{0,1\}^n| = 2^n = m_1 + 2m_2$ and the number we are looking for is $m_1 + m_2 = \frac{2m_1+2m_2}{2} = \frac{2^n+m_1}{2} = 2^{n-1} + \frac{m_1}{2}$. To find m_1, we must count the number of strings $x_1, \dots x_n$ with $x_1 = x_n$, $x_2 = x_{n-1}$, etc. If n is even, there are exactly $2^{n/2}$ such strings, since we can specify one by giving the first $n/2$ bits (which determine the rest uniquely). If n is odd, there are exactly $2^{(n+1)/2}$ such strings, since the middle bit can be set freely. We can write both alternatives as $2^{\lceil n/2 \rceil}$, giving $|\{0,1\}^n/\sim| = 2^{n-1} + 2^{\lceil n/2 \rceil}$.
2. In this case, observe that $x \approx y$ if and only if x and y contain the same number of 1 bits. There are $n+1$ different possible values $0, 1, \dots, n$ for this number. So $|\{0,1\}^n/\approx| = n + 1$.

The solution to the first part assumes $n > 0$; otherwise it produces the nonsensical result $3/2$. The problem does not specify whether $n = 0$ should be considered; if it is, we get exactly one equivalence class for both parts (the empty set).

2 Linear transformations (20 points)

Show whether each of the following functions from $\mathbb{R}^2$ to $\mathbb{R}$ is a linear transformation or not.

$$\begin{aligned} f_1(x) &= x_1 - x_2. \\ f_2(x) &= x_1 x_2. \\ f_3(x) &= x_1 + x_2 + 1. \\ f_4(x) &= \frac{x_1^2 - x_2^2 + x_1 - x_2}{x_1 + x_2 + 1}. \end{aligned}$$

Clarification added during the exam: You may assume that $x_1 + x_2 \neq -1$ for f_4.

Solution

1. Linear: $f_1(ax) = ax_1 - ax_2 = a(x_1 - x_2) = af_1(x)$ and $f_1(x + y) = (x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) = f_1(x) + f_1(y)$.
2. Not linear: $f_2(2x) = (2x_1)(2x_2) = 4x_1x_2 = 4f_2(x) \neq 2f_2(x)$ when $f_2(x) \neq 0$.
3. Not linear: $f_3(2x) = 2x_1 + 2x_1 + 1$ but $2f_3(x) = 2x_1 + 2x_2 + 2$. These are never equal.

4. Linear:

$$\begin{aligned}
f_4(x) &= \frac{x_1^2 - x_2^2 + x_1 - x_2}{x_1 + x_2 + 1} \\
&= \frac{(x_1 + x_2)(x_1 - x_2) + (x_1 - x_2)}{x_1 + x_2 + 1} \\
&= \frac{(x_1 + x_2 + 1)(x_1 - x_2)}{x_1 + x_2 + 1} \\
&= x_1 - x_2 \\
&= f_1(x).
\end{aligned}$$

Since we've already shown f_1 is linear, $f_4 = f_1$ is also linear.

A better answer is that f_4 is not a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}$ because it's not defined when $x_1 + x_2 - 1 = 0$. The clarification added during the exam tries to work around this, but doesn't really work. A better clarification would have defined f_4 as above for most x, but have $f_4(x) = x_1 - x_2$ when $x_1 + x_2 = -1$. Since I was being foolish about this myself, I gave full credit for any solution that either did the division or noticed the dividing-by-zero issue.

3 Flipping coins (20 points)

Flip n independent fair coins, and let X be a random variable that counts how many of the coins come up heads. Let a be a constant. What is $\mathrm{E}[a^X]$?

Solution

To compute $\mathrm{E}[a^X]$, we need to sum over all possible values of a^X weighted by their probabilities. The variable X itself takes on each value $k \in \{0 \dots n\}$ with probability $\binom{n}{k}2^{-n}$, so a^X takes on each corresponding value a^k with the same probability. We thus have:

$$\begin{aligned}
\mathrm{E}[a^X] &= \sum_{k=0}^{n} a^k \binom{n}{k} 2^{-n} \\
&= 2^{-n} \sum_{k=0}^{n} \binom{n}{k} a^k 1^{n-k} \\
&= 2^{-n}(a+1)^n \\
&= \left(\frac{a+1}{2}\right)^n.
\end{aligned}$$

The second to last step uses the Binomial Theorem.

As a quick check, some easy cases are $a = 0$ with $\mathrm{E}[a^X] = (1/2)^n$, which is consistent with the fact that $a^X = 1$ if and only if $X = 0$; and $a = 1$ with $\mathrm{E}[a^X] = 1^n = 1$, which is consistent with $a^X = 1^X = 1$ being constant. Another easy case is $n = 1$, in which we can compute directly $\mathrm{E}[a^X] = (1/2)a^0 + (1/2)a^1 = \frac{a+1}{2}$ as given by the formula. So we can have some confidence that we didn't mess up in the algebra somewhere.

Note also that $\mathrm{E}[a^X] = \left(\frac{a+1}{2}\right)^n$ is generally not the same as $a^{\mathrm{E}[X]} = a^{n/2}$.

4 Subtracting dice (20 points)

Let X and Y represent independent 6-sided dice, and let $Z = |X - Y|$ be their difference. (For example, if $X = 4$ and $Y = 3$, then $Z = 1$, and similarly when $X = 2$ and $Y = 5$, then $Z = 3$.)

1. What is $\Pr[Z = 1]$?
2. What is $\mathrm{E}[Z]$?
3. What is $\mathrm{E}[Z|Z \neq 0]$?

Solution

1. There are five cases where $Z = 1$ with $Y = X + 1$ (because X can range from 1 to 5), and five more cases where $Z = 1$ with $X = Y + 1$. So $\Pr[Z = 1] = \frac{10}{36} = \frac{5}{18}$.

2. Here we count 10 cases where $Z = 1$, 8 cases where $Z = 2$ (using essentially the same argument as above; here the lower die can range up to 4), 6 where $Z = 3$, 4 where $Z = 4$, and 2 where $Z = 5$. The cases where $Z = 0$ we don't care about. Summing up, we get $\mathrm{E}[Z] = (10 \cdot 1 + 8 \cdot 2 + 6 \cdot 3 + 4 \cdot 4 + 2 \cdot 5)/36 = 70/36 = 35/18$.

3. We can avoid recomputing all the cases by observing that $\mathrm{E}[Z] = \mathrm{E}[Z|Z \neq 0] \Pr[Z \neq 0] + \mathrm{E}[Z|Z = 0] \Pr[Z = 0]$. Since $\mathrm{E}[Z|Z = 0] = 0$, the second term disappears and we can solve for $\mathrm{E}[Z|Z \neq 0] = \mathrm{E}[Z]/\Pr[Z \neq 0]$. We can easily calculate $\Pr[Z = 0] = 1/6$ (since both dice are equal in this case, giving 6 out of 36 possible rolls), from which we get $\Pr[Z \neq 0] = 1 - \Pr[Z = 0] = 5/6$. Plugging this into our previous formula gives $\mathrm{E}[Z|Z \neq 0] = \frac{(35/18)}{(5/6)} = 7/3$.

It is also possible (and acceptable) to solve this problem by building a table of all 36 cases and summing up the appropriate values.

5 Scanning an array (20 points)

Suppose you have an $m \times m$ array in some programming language, that is, an data structure A holding a value $A[i, j]$ for each $0 \leq i < m$ and $0 \leq j < m$. You'd like to write a program that sets every element of the array to zero.

The usual way to do this is to start with $i = 0$ and $j = 0$, increment j until it reaches m, then start over with $i = 1$ and $j = 0$, and repeat until all m^2 elements of A have been reached. But this requires two counters. Instead, a clever programmer suggests using one counter k that runs from 0 up to $m^2 - 1$, and at each iteration setting $A[3k \bmod m, 7k \bmod m]$ to zero.

For what values of $m > 0$ does this approach actually reach all m^2 locations in the array?

Solution

Any two inputs k that are equal mod m give the same pair ($3k \bmod m, 7k \bmod m$). So no matter how many iterations we do, we only reach m distinct locations. This equals m^2 only if $m = 1$ or $m = 0$. The problem statement excludes $m = 0$, so we are left with $m = 1$ as the only value of m for which this method works.