CS202 Midterm Exam

October 24th, 2007

Write your answers on the exam. Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately 50 minutes to complete this exam.

1 Dueling recurrences (20 points)

Let $0 \leq S(0) \leq T(0)$, and suppose we have the recurrences

$$\begin{aligned} S(n+1) &= aS(n) + f(n) \\ T(n+1) &= bT(n) + g(n), \end{aligned}$$

where $0 \leq a \leq b$ and $0 \leq f(n) \leq g(n)$ for all $n \in \mathbb{N}$.

Prove that $S(n) \leq T(n)$ for all $n \in \mathbb{N}$.

Solution

We'll show the slightly stronger statement $0 \leq S(n) \leq T(n)$ by induction on n. The base case $n = 0$ is given.

Now suppose $0 \leq S(n) \leq T(n)$; we will show the same holds for $n + 1$. First observe $S(n+1) = aS(n)+f(n) \geq 0$ as each variable on the right-hand side is non-negative. To show $T(n + 1) \geq S(n + 1)$, observe

$$\begin{aligned} T(n+1) &= bT(n) + g(n) \\ &\geq aT(n) + f(n) \\ &\geq aS(n) + f(n) \\ &= S(n+1). \end{aligned}$$

Note that we use the fact that $0 \leq T(n)$ (from the induction hypothesis) in the first step and $0 \leq a$ in the second. The claim does not go through without these assumptions, which is why using $S(n) \leq T(n)$ by itself as the induction hypothesis is not enough to make the proof work.

2 Seating arrangements (20 points)

A group of k students sit in a row of n seats. The students can choose whatever seats they wish, provided: (a) from left to right, they are seated in alphabetical order; and (b) each student has an empty seat immediately to his or her right.

For example, with 3 students `A`, `B`, and `C` and 7 seats, there are exactly 4 ways to seat the students: `A-B-C--`, `A-B--C-`, `A--B-C-`, and `-A-B-C-`.

Give a formula that gives the number of ways to seat k students in n seats according to the rules given above.

Solution

The basic idea is that we can think of each student and the adjacent empty space as a single width-2 unit. Together, these units take up $2k$ seats, leaving $n - 2k$ extra empty seats to distribute between the students. There are a couple of ways to count how to do this.

Combinatorial approach

Treat each of the k student-seat blocks and $n - 2k$ extra seats as filling one of $k + (n - 2k) = n - k$ slots. There are exactly $\binom{n-k}{k}$ ways to do this.

Generating function approach

Write $z + z^2$ for the choice between a width-1 extra seat and a width-2 student-seat block. For a row of $n - k$ such things, we get the generating function $(z + z^2)^{n-k} = z^{n-k}(1 + z)^{n-k} = z^{n-k} \sum_{i=0}^{n-k} \binom{n-k}{i} z^i = \sum_{i=0}^{n-k} \binom{n-k}{i} z^{n-k+i}$. The z^n coefficient is obtained when $i = k$, giving $\binom{n-k}{k}$ ways to fill out exactly n seats.

3 Non-attacking rooks (20 points)

Place n rooks at random on an $n \times n$ chessboard (i.e., an $n \times n$ grid), so that all $\binom{n^2}{n}$ placements are equally likely. What is the probability of the event that every row and every column of the chessboard contains exactly one rook?

Solution

We need to count how many placements of rooks there are that put exactly one rook per row and exactly one rook per column. Since we know that there is one rook per row, we can specify where these rooks go by choosing a unique column for each row. There are n choices for the first row, $n - 1$

remaining for the second row, and so on, giving $n(n-1)\cdots 1 = n!$ choices altogether. So the probability of the event is $n!/\binom{n^2}{n} = (n^2-n)!/(n^2)!$.

4 Subsets (20 points)

Let $A \subseteq B$.

1. Prove or disprove: There exists an injection $f : A \to B$.
2. Prove or disprove: There exists a surjection $g : B \to A$.

Solution

1. Proof: Let $f(x) = x$. Then $f(x) = f(y)$ implies $x = y$ and f is injective.
2. Disproof: Let B be nonempty and let $A = \emptyset$. Then there is no function at all from B to A, surjective or not.