# CS202 Midterm Exam

October 21st, 2010

Write your answers on the exam. Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately 75 minutes to complete this exam.

## 1 A partial order (20 points)

Let $S \subseteq \mathbb{N}$, and for any $x, y \in \mathbb{N}$, define $x \preceq y$ if and only if there exists $z \in S$ such that $x + z = y$.

Show that if $\preceq$ is a partial order, then (a) 0 is in $S$ and (b) for any $x, y$ in $S$, $x + y$ is in $S$.

### Solution

If $\preceq$ is a partial order, then by reflexivity we have $x \preceq x$ for any $x$. But then there exists $z \in S$ such that $x + z = x$, which can only happen if $z = 0$. Thus $0 \in S$.

Now suppose $x$ and $y$ are both in $S$. Then $0 + x = x$ implies $0 \preceq x$, and $x + y = x + y$ implies $x \preceq x + y$. Transitivity of $\preceq$ gives $0 \preceq x + y$, which occurs only if some $z$ such that $0 + z = x + y$ is in $S$. The only such $z$ is $x + y$, so $x + y$ is in $S$.

## 2 Big exponents (20 points)

Let $p$ be a prime, and let $0 \leq a < p$. Show that $a^{2p-1} = a \pmod{p}$.

### Solution

Write $a^{2p-1} = a^{p-1}a^{p-1}a$. If $a \neq 0$, Euler's Theorem (or Fermat's Little Theorem) says $a^{p-1} = 1 \pmod{p}$, so in this case $a^{p-1}a^{p-1}a = a \pmod{p}$. If $a = 0$, then (since $2p - 1 \neq 0$), $a^{2p-1} = 0 = a \pmod{p}$.

## 3 At the playground (20 points)

Let $L(x, y)$ represent the statement "$x$ likes $y$" and let $T(x)$ represent the statement "$x$ is tall," where $x$ and $y$ range over a universe consisting of all children on a playground. Let $m$ be "Mary," one of the children.

1. Translate the following statement into predicate logic: "If $x$ is tall, then Mary likes $x$ if and only if $x$ does not like $x$."

2. Show that if the previous statement holds, Mary is not tall.

### Solution

1. $\forall x\,(T(x) \Rightarrow (L(m,x) \Leftrightarrow \neg L(x,x)))$.

2. Suppose the previous statement is true. Let $x = m$, then $T(m) \Rightarrow (L(m,m) \Leftrightarrow \neg L(m,m))$. But $L(m,m) \Leftrightarrow \neg L(m,m)$ is false, so $T(m)$ must also be false.

## 4 Gauss strikes back (20 points)

Give a closed-form formula for $\sum_{k=a}^{b} k$, assuming $0 \le a \le b$.

### Solution

Here are three ways to do this:

1. Write $\sum_{k=a}^{b} k$ as $\sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k$ and then use the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ to get

$$\begin{aligned}\sum_{k=a}^{b} k &= \sum_{k=1}^{b} k - \sum_{k=1}^{a-1} k \\ &= \frac{b(b+1)}{2} - \frac{(a-1)a}{2} \\ &= \frac{b(b+1) - a(a-1)}{2}.\end{aligned}$$

2. Use Gauss's trick, and compute

$$\begin{aligned}2\sum_{k=a}^{b} k &= \sum_{k=a}^{b} k + \sum_{k=a}^{b} (b+a-k) \\ &= \sum_{k=a}^{b} (k+b+a-k) \\ &= \sum_{k=a}^{b} (b+a) \\ &= (b-a+1)(b+a).\end{aligned}$$

Dividing both sides by 2 gives $\frac{(b-a+1)(b+a)}{2}$.

3. Write $\sum_{k=a}^{b} k$ as $\sum_{k=0}^{b-a}(a+k) = (b-a+1)a + \sum_{k=0}^{b-a} k$. Then use the sum formula as before to turn this into $(b-a+1)a + \frac{(b-a)(b-a+1)}{2}$.

Though these solutions appear different, all of them can be expanded to $\frac{b^2-a^2+a+b}{2}$.