Sorting

$i = 0$
$i = 2$

```
def insertion_sort(l):
    n = len(l)
    i = 1
    while i < n:
        j = i
        while j - 1 >= 0 and l[j] < l[j - 1]:
            temp = l[j - 1]
            l[j - 1] = l[j]
            l[j] = temp
            j = j - 1
        i = i + 1
    return l
```

get to this line n times

	assign	array	len	add	sub	compare	and
`n = len(l)`	1		1				
`i = 1`	1						
`while i < n:`						n	
`j = i`	$n-1$				$n-1$		
`while j - 1 >= 0 and l[j] < l[j - 1]:`		n^2+n-2			n^2+n-2	n^2+n-2	n^2+n-2
`temp = l[j - 1]` … `j = j - 1`	$2n^2-2n$	$2n^2-2n$			$\frac{3}{2}n^2-\frac{3}{2}n$		
`i = i + 1`	$n-1$			$n-1$			
TOTAL OPS	$2n^2$	$3n^2-n-2$	1	$n-1$	$\frac{5}{2}n^2+\frac{1}{2}n-3$	n^2+2n-2	n^2+n-2
time/op	t_1	t_2	t_3	t_4	t_5	t_6	t_7

there may be mistakes in here, but the point is that our tool for comparing running times shouldn't require us to do this much work

TOTAL TIME $(2t_1 + 3t_2 + \frac{5}{2}t_5 + t_6 + t_7)n^2 - (t_2 - t_4 - \frac{1}{2}t_5 - 2t_6 - t_7)n - (2t_2 - t_3 + t_4 + 3t_5 + 2t_6 + 2t_7)$

informal: "insertion sort is $O(n^2)$"

formal + informative: "the worst-case running time of insertion sort is $\Theta(n^2)$"

```
def selection_sort(l):
    n = len(l)
    i = 0
    while i < n - 1:
        smallest_loc = i
        j = i + 1
        while j < n:
            if l[j] < l[smallest_loc]:
                smallest_loc = j
            j = j + 1
        temp = l[smallest_loc]
        l[smallest_loc] = l[i]
        l[i] = temp
        i = i + 1
    return l
```

	assign	array	len	add	sub	comp
`n = len(l)`	1		1			
`i = 0`	1				n	n
`smallest_loc = i`	$n-1$					
`j = i + 1`	$n-1$			$n-1$		
`while j < n:`						$\frac{1}{2}n^2+\frac{1}{2}n-1$
`if l[j] < l[smallest_loc]:`		$2n^2-2n$				$\frac{1}{2}n^2-\frac{1}{2}n$
`smallest_loc = j`	$\frac{1}{2}n^2-\frac{1}{2}n$					
`j = j + 1`	$\frac{1}{2}n^2-\frac{1}{2}n$			$\frac{1}{2}n^2-\frac{1}{2}n$		
`temp = l[smallest_loc]` … `i = i + 1`	$4n-4$	$4n-4$		$4n-4$		

TOTAL TIME $(t_1 + 2t_2 + \frac{1}{2}t_4 + t_6)n^2 + (5t_1 - 2t_2 + \frac{9}{2}t_4 + t_5 + t_6)n - (4t_1 + 4t_2 - t_3 + 5t_4 + t_6)$

$f \leq g$

size of input to alg

time used
steps
storage used

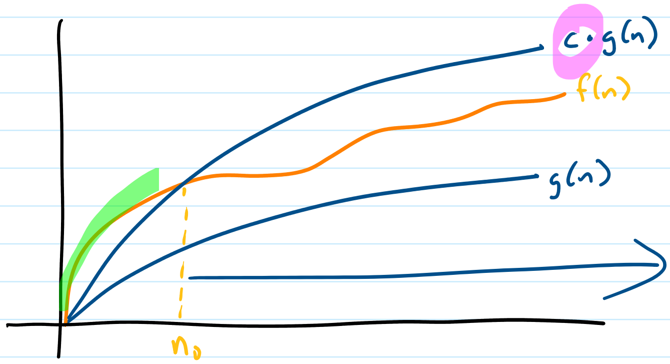
DEF: For $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$

f is big-Oh of g

f is order at most g ($f(n) \in O(g(n))$) means

$f \leq g$

$\exists n_0 \in \mathbb{Z}^+, c \in \mathbb{R}^+$ s.t. $\forall n \geq n_0$, $f(n) \leq c \cdot g(n)$



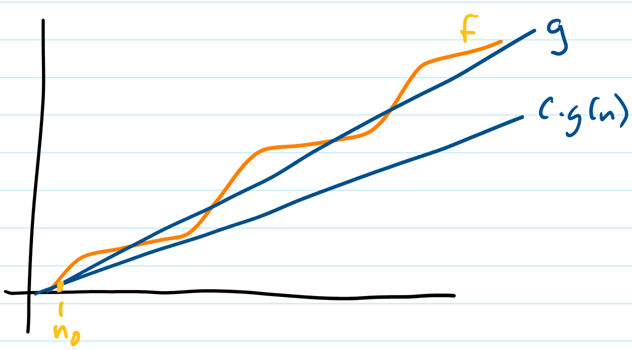
$\frac{1}{2} \cdot n^2 \in O\left(\frac{1}{20000000} \frac{1.62^n}{\sqrt{5}}\right)$

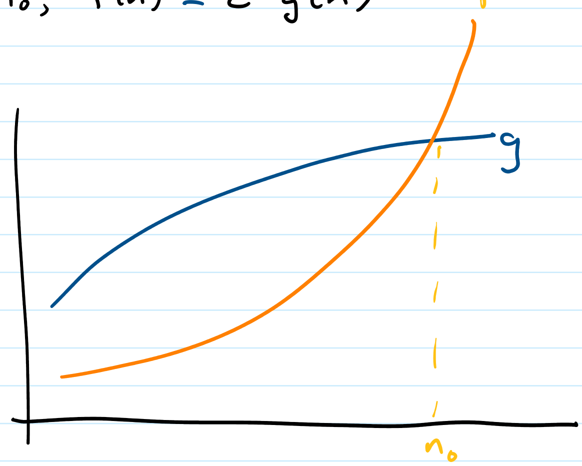
$n^2 \in O(1.62^n)$

f is order at least g ($f(n) \in \Omega(g(n))$) means

$\exists n_0 \in \mathbb{Z}^+, c \in \mathbb{R}^+$ s.t. $\forall n \geq n_0$, $f(n) \geq c \cdot g(n)$

$f \geq g$



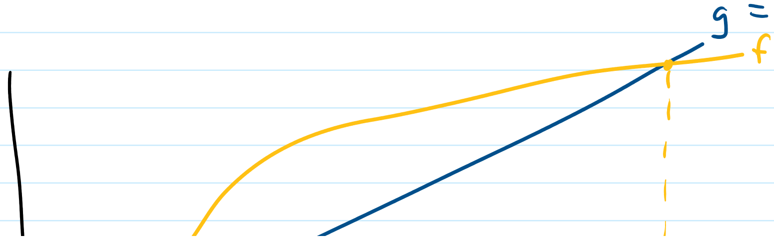


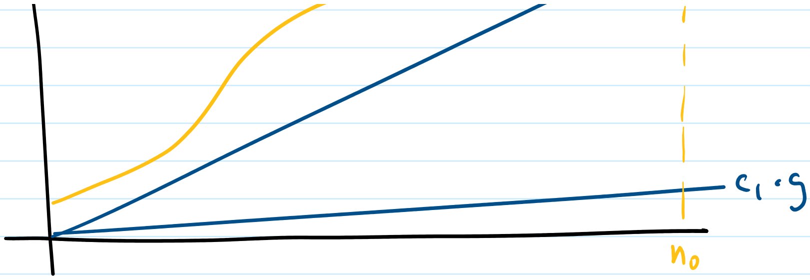
$f = g$

f is order g ($f(n) \in \Theta(g(n))$) means

$\exists n_0 \in \mathbb{Z}^+, c_1, c_2 \in \mathbb{R}^+$ s.t. $\forall n \geq n_0$, $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$

$g = 1 \cdot g = c_2 \cdot g$




$c_1 \cdot g$
n_0

Examples

$\underbrace{5n+3}_{f} \in O(\underbrace{n}_{g})$

Need: $\exists n_0 \in \mathbb{Z}^+, c \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+\ n \geq n_0 \rightarrow 5n+3 \leq c \cdot n$

For which n is $5n+3 \leq 6n$? $n \geq 3$

Let $c = 6$ and $n_0 = 3$.

Suppose $n \in \mathbb{Z}^+$ with $n \geq 3$

Then $5n+n \geq 5n+3$

$5n+3 \leq 6n$

So $\forall n \in \mathbb{Z}^+, n \geq 3 \rightarrow 5n+3 \leq 6n$

$\therefore \exists n_0 \in \mathbb{Z}^+$ and $c \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+, n \geq n_0 \rightarrow 5n+3 \leq c \cdot n$

namely 3 namely 6

so $5n+3 \in O(n)$

$5n+3 \in \Omega(n)$

Need: $\exists n_0 \in \mathbb{Z}^+, c \in \mathbb{R}^+$ s.t. $\forall n \geq n_0,\ 5n+3 \geq c \cdot n$

$5 > 1$ so $5n > n$

$3 > 0$ $5n+3 > 5n$

for all $n \geq 1$, $5n+3 \geq n$

So $\exists n_0, c$ s.t. $\forall n \geq n_0,\ 5n+3 \geq c \cdot n$

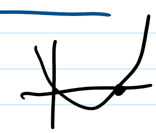
namely 1, 1

$5n+3 \in \Theta(n)$

$\exists n_0, c_1, c_2$ s.t. $\forall n \geq n_0\ \ c_1 \cdot n \leq 5n+3 \leq c_2 \cdot n$

(Ω … O)

Choose $c_1 = 1, c_2 = 6, n_0 = 3$



$\underbrace{33n+8}_{f} \in \underbrace{O(n^2)}_{g}$

want: $\exists n_0, c$ s.t. $\forall n \geq n_0\ \ 33n+8 \leq c \cdot n^2$

if $c = 2$

and $33n \leq n^2$ $33 \leq n$

and $8 \leq n^2$ $3 \leq n$

then $33n+8 \leq 2n^2$

Let $n \geq 33$ then $n^2 \geq 33n$

also $n \geq 3$ $n^2 \geq 9$

So $2n^2 \geq 33n+9 \geq 33n+8$

So $\forall n \geq 33,\ 33n+8 \leq 2n^2$

So $\exists n_0, c$ s.t. $\forall n \geq n_0,\ 33n+8 \leq c \cdot n^2$

namely 33, 2

$\therefore 33n+8 \in O(n^2)$

$2n^2 \in O(n^2-10n-3)$

want! $\exists n_0, c$ s.t. $\forall n \geq n_0\ \ 2n^2 \leq c \cdot (n^2 - 10n - 3)$

$2n^2 \leq 4n^2 - 40n - 12$

$n^2 \geq 40n$ $n \geq 40$

$n^2 \geq 12$ $n \geq \sqrt{12}$

$2n^2 \geq 40n + 12$

$4n^2 \geq 2n^2 + 40n + 12$

$4n^2 - 40n - 12 \geq 2n^2$

Let $n \geq 40$. Then $n^2 \geq 40n$

also, $n \geq \sqrt{12}$ so $n^2 \geq 12$

so $2n^2 \geq 40n + 12$

and $4n^2 \geq 2n^2 + 40n + 12$

and $4n^2 - 40n - 12 \geq 2n^2$

So $\forall n \geq 40,\ 2n^2 \leq 4(n^2 - 10n - 3)$

So $\exists n_0, c$ s.t. $\forall n \geq n_0,\ 2n^2 \leq c \cdot (n^2 - 10n - 3)$

(namely 40, 4)

so $2n^2 \in O(n^2 - 10n - 3)$

$33n+8 \notin \Omega(n^2)$

$33n+8 \in \Omega(n^2) \equiv \exists n_0, c$ s.t. $\forall n \geq n_0,\ 33n+8 \geq c \cdot n^2$

$33n+8 \notin \Omega(n^2) \equiv \forall n_0, c\ \exists n \geq n_0\ \ 33n+8 < c \cdot n^2$

Suppose $n_0 \in \mathbb{Z}^+$ and $c \in \mathbb{R}^+$ [want to find $n \geq n_0$ s.t.

$33n+8 \notin \Omega(n^2) \equiv \forall n_0, c\ \exists n \geq n_0\ \ 33n+8 < c \cdot n^2$

Suppose $n_0 \in \mathbb{Z}^+$ and $c \in \mathbb{R}^+$ [want to find $n \geq n_0$ s.t. $33n+8 < c \cdot n^2$]

let $n = \max(\lceil \frac{34}{c} \rceil, 8, n_0) + 1$

So $n > \frac{34}{c}$ so $cn^2 > 34n$ — $33n < cn^2$, $\frac{34}{c} < n$, $34n > 33n+8$

and $n > 8$ so $34n > 33n+8$ — $n > 8$

$\therefore cn^2 > 33n+8$

$\therefore \exists n \geq n_0$ s.t. $33n+8 < c \cdot n^2$ — ceiling ($\lceil x \rceil$ = the smallest integer n s.t. $x \leq n$)

↳ namely $n = \max(\lceil \frac{34}{c} \rceil, 8, n_0) + 1$

$\therefore \forall n_0, c, \exists n \geq n_0$ s.t. $33n+8 < cn^2$

THM: Let $f: \mathbb{Z}^+ \to \mathbb{R}^+$ be $f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 \cdot n + c_0$ where $c_d \in \mathbb{R}^+, d \in \mathbb{N}$

Then $f(n) \in \Theta(n^d)$

THM: Let $f: \mathbb{Z}^+ \to \mathbb{R}^+$ be $f(n) = c_k n^{p_k} + c_{k-1} n^{p_{k-1}} + \cdots + c_0 n^{p_0}$ where $c_k \in \mathbb{R}^+$, $p_k > p_{k-1} > \cdots > p_0$, $p_k, \ldots, p_0 \in \mathbb{R}^+$

Then $f(n) \in \Theta(n^{p_k})$

$4n^{2.8} + 6n^{2.1} + 9n^{1.5} - 16n + 8 \in \Theta(n^{2.8})$

$2^n \in O(3^n)$

but not $2^n \in \Omega(3^n)$

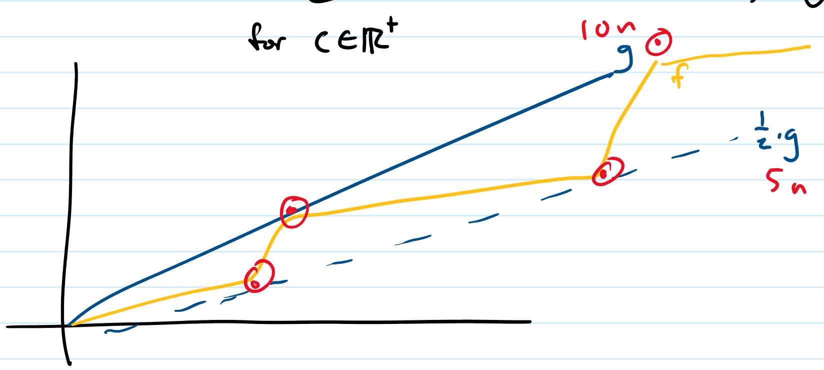
$$\lim_{n \to \infty} \frac{3^n}{2^n} = \lim_{n \to \infty} \left(\frac{3}{2}\right)^n = \infty$$

If $\displaystyle\lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty$ then $f(n) \in O(g(n))$

$= 0$ then $f(n) \in \Omega(g(n))$

$= c$ then $f(n) \in \Theta(g(n))$

for $c \in \mathbb{R}^+$



$f(n) \in \Theta(g(n))$

but $\displaystyle\lim_{n \to \infty} \frac{f(n)}{g(n)}$ does not exist

$\ln n \in O(\sqrt{n})$

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2} n^{-\frac{1}{2}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = 0$$

$\ln n \in O(n^{\frac{1}{1000}})$

$(\ln n)^{10} \in O(\underline{n^{\frac{1}{1000000}}})$

Properties

$2^{100} n^4 \in \Theta(n^4)$ $\quad$ $n^4 \in \Theta(2^{100} \cdot n^4)$

THM: For any function $f: \mathbb{Z}^+ \to \mathbb{R}^+$, $f(n) \in \Theta(f(n))$

$\forall n \geq 1,\ 1 \cdot f(n) \leq f(n) \leq 1 \cdot f(n)$

THM: For any functions $f, g: \mathbb{Z}^+ \to \mathbb{R}^+$, if $f(n) \in \Theta(g(n))$ then $g(n) \in \Theta(f(n))$

Proof: Let f, g be as given.

Then $\exists n_0 \in \mathbb{Z}^+, c_1, c_2 \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+, n \geq n_0 \to c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ (def Θ)

Find such n_0, c_1, c_2; let $n \geq n_0$ and $n \in \mathbb{Z}^+$. Then $c_1 \cdot g(n) \leq f(n)$ and $f(n) \leq c_2 \cdot g(n)$ (universal modus ponens)

and $g(n) \leq \frac{1}{c_1} \cdot f(n)$ and $\frac{1}{c_2} f(n) \leq g(n)$ (dividing by positives)

so $\frac{1}{c_2} \cdot f(n) \leq g(n) \leq \frac{1}{c_1} \cdot f(n)$

$\therefore \forall n \in \mathbb{Z}^+, n \geq \underset{m_0}{n_0} \to \underbrace{\frac{1}{c_2}}_{d_1} \cdot f(n) \leq g(n) \leq \underbrace{\frac{1}{c_1}}_{d_2} \cdot f(n)$ (generic particular + conditional)

$\therefore \exists m_0, d_1, d_2$ s.t. $\forall n \in \mathbb{Z}^+, n \geq m_0 \to d_1 \cdot f(n) \leq g(n) \leq d_2 \cdot f(n)$ (example)

($m_0 \in \mathbb{N}^+$, $d_1, d_2 \in \mathbb{R}^+$) ↳ namely $m_0 = n_0$, $d_1 = \frac{1}{c_2}$, $d_2 = \frac{1}{c_1}$

$\therefore g(n) \in \Theta(f(n))$ (def Θ)

THM: For any functions $f, g, h: \mathbb{Z}^+ \to \mathbb{R}^+$, if $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$

Proof: Let f, g, h be as given.

Then $\exists n_0, c$ s.t. $\forall n \geq n_0$, $f(n) \leq c \cdot g(n)$

and $\exists m_0, d$ s.t. $\forall n \geq m_0$, $g(n) \leq d \cdot h(n)$

[want $\exists k_0, e$ s.t. $\forall n \geq k_0$, $f(n) \leq e \cdot h(n)$]

Let $n \geq \max(n_0, m_0)$, $n \in \mathbb{Z}^+$.

Then $n \geq n_0$ and $n \geq m_0$ so $\frac{1}{c} \cdot f(n) \leq g(n) \leq d \cdot h(n)$

$\frac{1}{c} \cdot f(n) \leq d \cdot h(n)$

$f(n) \leq c \cdot d \cdot h(n)$

$e \in \mathbb{R}^+$ b/c $c, d \in \mathbb{R}^+$

$\forall n \in \mathbb{Z}^+, n \geq \underbrace{\max(n_0, m_0)}_{k_0} \to f(n) \leq \underbrace{c \cdot d}_{e} \cdot h(n)$

$\exists k_0 \in \mathbb{Z}^+, e \in \mathbb{R}^+$ s.t. $\forall n \in \mathbb{Z}^+, n \geq \max(n_0, m_0) \to f(n) \leq c \cdot d \cdot h(n)$

$f(n) \in O(h(n))$

THM: For all $f, g, h: \mathbb{Z}^+ \to \mathbb{R}^+$, if $f(n) \in O(g(n))$ then $g(n) \in \Omega(f(n))$

if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$ then $f(n) \in \Theta(g(n))$

$f \, R \, g$ when $f(n) \in \Theta(g(n))$

equivalence relations

if $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$ then $f(n) \in \Theta(h(n))$