TEST SETS FOR INTEGER PROGRAMS, $\forall\ \exists$ SENTENCES

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1 Introduction

Suppose A is a fixed $m \times n$ matrix of integers of rank n . Let b be a vector of "right hand sides" that varies over $\mathbf{R}^m$ and consider the polyhedron $K_b = \{x \in \mathbf{R}^n : Ax \leq b\}$. From basic theorems in linear programming, we know that if K_b is nonempty, then it contains a vertex. From this, it follows that there are at most m^n matrices $T_1, T_2, \ldots$, each of which is $n \times m$ so that for all $b \in \mathbf{R}^m$ with $K_b \neq \emptyset$, one of the $T_i b$ belongs to K_b. We may call $T_1 b, T_2 b, \ldots$ a "test set" for linear programming. Note three properties of the test set

- Each function T_i is an affine function that maps b to a candidate point in $\mathbf{R}^n$.
- For fixed n, the number of affine functions $T_1, T_2, \ldots$ is bounded above by a polynomial.
- For fixed n, the affine functions can be computed in polynomial time from A.

Theorem (3.1) of this paper will prove an analog for Integer Programming. The second part of section 3 uses this theorem to derive a decision procedure (which is polynomial time bounded for fixed p, n) to decide the truth / falsity of any sentence of the following form (where Q is a given copolyhedron in $\mathbf{R}^p$ and A, B, b are given matrices of suitable dimensions. See Notation below.)

$$\forall y \in Q/\mathbf{Z}^l \quad \exists x \in \mathbf{Z}^n \quad Ax + By \leq b.$$

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The sentence says in words : for every y for which an integer vector $z \in \mathbf{Z}^l$ exists such that (y, z) is in Q, there exists also an integer vector x so that $Ax + By \leq b$.

The first theorem is proved as follows : In [8], Theorem (1.1) below is proved which describes the structure of $K_b + \mathbf{Z}^n$. (See notation below) as b varies over a **bounded** set in $\mathbf{R}^m$. The proof of this theorem involved several recent developments in the Geometry of Numbers. This note is an extension and application of Theorem (1.1). First, Theorem (1.1) is extended to the case when b varies over all of $\mathbf{R}^m$, not just a bounded subset of it. The extension is Theorem (2.1). Then the test set theorem is (3.1). The decision procedure for $\forall\exists$ sentences is described in Theorem (3.2).

Notation

$\mathbf{R}^n$ is Euclidean n space. The lattice of all integer vectors in $\mathbf{R}^n$ is denoted $\mathbf{Z}^n$. For any two sets $S, T \subseteq \mathbf{R}^n$, we denote by $S + T$ the set $\{s + t : s \in S; t \in T\}$. For any positive real, λ, we denote by λS, the set $\{\lambda s : s \in S\}$. For any set W in $\mathbf{R}^{n+l}$ and any set V in $\mathbf{R}^l$, we denote by W/V the set

$$\{x : x \in \mathbf{R}^n \text{ such that there exists a } y \in V \text{ with } (x, y) \in W\}.$$

W/V is the set obtained by "projecting out" V from W.

A **copolyhedron** is the intersection of a finite number of half spaces - some of them closed and the others open. ("co" for closed / open.) If a copolyhedron is bounded, I will call it a copolytope.

Some statements in the paper will assert "the algorithm *finds* copolytope P_i....." . The precise meaning of this statement is as follows : suppose P_i is in $\mathbf{R}^n$. The algorithm will find a rational $m \times (n+l)$ matrix C and a rational $m \times 1$ vector b where l is at most some polynomial function of n and for each row of A, either the $\leq$ or the $<$ sign such that P_i equals

$$\{x : x \in \mathbf{R}^n \text{ such that there exists a } y \in \mathbf{R}^l \text{ with } C\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \leq \\ < \end{pmatrix} b\}.$$

In much of the paper A will be a fixed $m \times n$ matrix. If the meaning of A is clear from the context, for any b in $\mathbf{R}^m$, the polyhedron $\{x \in \mathbf{R}^n : Ax \leq b\}$ will be denoted by K_b. In much of the paper, b will vary over some

copolyhedron in $\mathbf{R}^m$. Some bounds in the paper will be in terms of the affine dimension j_o of this copolyhedron. The "size" of a rational matrix is the number of bits needed to express it. It is assumed that integers are written in binary notation, so it takes a $O(\log M)$ length string to express an integer of magnitude M.

A basis B of the lattice $\mathbf{Z}^n$ is a set of n linearly independent vectors $\{b_1, b_2, \ldots b_n\}$ in $\mathbf{Z}^n$, such that each member of $\mathbf{Z}^n$ can be expressed as an integer linear combination of $\{b_1, b_2, \ldots b_n\}$. The "fundamental parallelopied" corresponding to B is the set $\{x : x = \sum_{i=1}^{n} \lambda_i b_i$ where $\lambda_i \in \mathbf{R}$ satisfy $0 \leq \lambda_i < 1\}$. It is denoted $F(B)$. For each point y in $\mathbf{R}^n$, there is a unique lattice point z such that $z+F(B)$ contains y. The parallelopied $z+F(B)$ is denoted $F(B; y)$.

Theorem (1.1)[8] Let A be an $m \times n$ matrix of integers of size ϕ. Let P be a copolytope in $\mathbf{R}^m$ of affine dimension j_o such that for all $b \in P$, the set $K_b = \{x : Ax \leq b\}$ is nonempty and bounded. Let $M = (\max_{b \in P}(|b| + 1))$. There is an algorithm which for any fixed n, j_o runs in time polynomial in the size of $\phi, \log M$ and and finds a partition of $P \times \mathbf{R}^n$ into subsets $S_1, S_2, \ldots S_r$ such that :

1. $r \leq (n\phi m \log M)^{j_o n^{dn}}$, where d is a constant independent of n, m, M, ϕ.
2. Each S_i is of the form $S_i'/\mathbf{Z}^l$ where S_i' is a copolyhedron in $\mathbf{R}^{m+n+l}$ and $l \leq n^{3n}$.
3. Letting $S_i(b) = \{x \in \mathbf{R}^n : (b, x) \in S_i\}$, we have for all i and all $b \in P$, $S_i(b) + \mathbf{Z}^n = S_i(b)$.

The algorithm also finds corresponding to each S_i, a collection $\mathcal{B}_i$ of at most n^{3n} bases of $\mathbf{Z}^n$. Corresponding to each basis B in each $\mathcal{B}_i$, it finds an affine transformation $T(B) : \mathbf{R}^m \to \mathbf{R}^n$ and a set $Z(B)$ of at most n^n points of $\mathbf{Z}^n$ such that for all i and all $b \in P$, we have

$$(K_b + \mathbf{Z}^n) \cap S_i(b) =$$
$$\left[\left\{ \bigcup_{B \in \mathcal{B}_i} ((K_b + Z(B)) \cap F(B; T(B)b)) \right\} + \mathbf{Z}^n \right] \cap S_i(b).$$

2 The case of unbounded right hand sides

This section proves Theorem (2.1) which extends Theorem (1.1) to the case when P is a copolyhedron. In this case, the parameter ϕ, the size of the matrix A will essentially substitute $\log(\max_{b \in P})|b|$. Here is a precise statement of the theorem.

Theorem (2.1) Let A be an $m \times n$ matrix of integers of size ϕ with the property that $\{x : Ax \leq 0\} = \{0\}$ (or equivalently, K_b is bounded for all b). Let P be a copolyhedron in $\mathbf{R}^m$ of affine dimension j_o such that for all $b \in P$, the set $K_b = \{x : Ax \leq b\}$ is nonempty. There is an algorithm which for any fixed n, j_o runs in time polynomial in the size of the input and finds subsets a partition of $P \times \mathbf{R}^n$ into subsets $R_1, R_2, \ldots R_r$ such that :

1. $r \leq (n\phi m)^{j_o n^{en}}$ where e is a constant.
2. Each R_i is of the form $R_i'/\mathbf{Z}^l$ where R_i' is a copolyhedron in $\mathbf{R}^{m+n+l}$ and $l \leq n^{3n}$.
3. Letting $R_i(b) = \{x \in \mathbf{R}^n : (b, x) \in R_i\}$, we have for all i and all $b \in P$, $R_i(b) + \mathbf{Z}^n = R_i(b)$.

The algorithm also finds corresponding to each R_i, a collection $\mathcal{B}_i$ of at most n^{3n} bases of $\mathbf{Z}^n$. Corresponding to each basis B in each $\mathcal{B}_i$, it finds an affine transformation $T(B) : \mathbf{R}^m \to \mathbf{R}^n$ and a set $Z(B)$ of at most n^n points of $\mathbf{Z}^n$ such that for all i and all $b \in P$, we have

$$(K_b + \mathbf{Z}^n) \cap R_i(b) =$$

$$\left[\left\{ \bigcup_{B \in \mathcal{B}_i} ((K_b + Z(B)) \cap F(B; T(B)b)) \right\} + \mathbf{Z}^n \right] \cap R_i(b).$$

/*END OF STATEMENT OF THE THEOREM*/

I will prove the theorem by using Theorem (1.1). To do so, I will show using lemma (2.2) below that for any $b \in P$, the description of $K_b + \mathbf{Z}^n$ can be easily obtained from the description of $K_c + \mathbf{Z}^n$ where c has all its components in the range $[0 \;\; n2^{3\phi}]$. Further, I will show that c is a "piecewise

affine" function of b ; i.e., that P can be partitioned into polynomially many copolyhedra such that for each copolyhedron in the partition, there is an affine function that maps b to c. This proof will use lemma (2.3). Throughout this section, I let M denote $n2^{3\phi}$.

Lemma (2.2) : Let A be an $m \times n$ matrix of integers of size ϕ. Suppose b is any point in $\mathbf{R}^m$ with $b \geq 0$. (So, 0 is in K_b.) Define $b' = (b'_1, b'_2, \ldots b'_m)$ by : $b'_i = \min\{b_i, n2^{3\phi}\}$. Then,

$$K_b + \mathbf{Z}^n = K_{b'} + \mathbf{Z}^n.$$

Proof : The proof is based on the following fact due to Cook, Gerards, Schrijver and Tardos [3] : Let Δ be the maximum absolute value of any subdeterminant of A. If a point p belongs to K_b, and if K_b contains some point in $\mathbf{Z}^n$, then there is a point $q \in \mathbf{Z}^n \cap K_b$ with $|p_i - q_i| \leq n\Delta$ for $i = 1, 2, \ldots n$. (This fact is true for any "right hand side" b.)

It is clear that

$$K_b + \mathbf{Z}^n \supseteq K_{b'} + \mathbf{Z}^n.$$

Now, I will prove the converse. Suppose x is any point in $K_b + \mathbf{Z}^n$. Then $K_b - x$ contains an integer point; it also contains $-x$. So, by the above fact, there is an integer point z in $K_b - x$ with $|z_i + x_i| \leq n\Delta$ for all i. By Theorem 3.2 of [21], Δ is at most $2^{2\phi}$. It is now easy to see that $z + x$ belongs to $K_{b'}$ finishing the proof of the lemma.

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Suppose $v \cdot x = v_o$ is a hyperplane in Euclidean space. It partitions space into two "regions" - $\{x : v \cdot x \leq v_o\}$ and $\{x : v \cdot x > v_o\}$. Similarly, a set of l hyperplanes in $\mathbf{R}^m$ partition $\mathbf{R}^m$ into (at most) 2^l "regions" each region being determined by which side of each hyperplane it is on. There is another well-known upper bound on the number of regions - it is

$$\sum_{k=0}^{m} \binom{l}{k}.$$

For $l \leq m$, the sum is 2^l and the result is obvious. For $l > m$, we proceed by induction. The number of regions formed by the first $l-1$ of the hyperplanes

is at most $\sum_{k=0}^{m} \binom{l-1}{k}$ by induction. Now imagine adding the l th hyperplane. I claim that the number of existing regions that the l th hyperplane intersects is at most $\sum_{k=0}^{m-1} \binom{l-1}{k}$ - to see this, note that the intersections of the existing regions with the l th hyperplane form a partition of the l th hyperplane (an $m-1$ dimensional affine space). Each region intersected by the l th hyperplane is divided into two by it. So we get the total number of regions formed by all the l hyperplanes is at most

$$\sum_{k=0}^{m} \binom{l-1}{k} + \sum_{k=0}^{m-1} \binom{l-1}{k} = \sum_{k=1}^{m} \left(\binom{l-1}{k-1} + \binom{l-1}{k} \right) + 1$$

which proves the claim. The lemma below follows immediately.

Lemma (2.3) Suppose V is a j dimensional affine subspace of $\mathbf{R}^m$. For any set of l hyperplanes in $\mathbf{R}^m$, the number of regions in the partition of $\mathbf{R}^m$ by the l hyperplanes that V intersects is at most

$$\sum_{k=0}^{j} \binom{l}{k} \leq l^j.$$

Further, if j is fixed, then given the hyperplanes and V, we can find the regions intersected by V in polynomial time.

Proof : The first part is already proved. For the algorithm, we go again to the first part of the proof and see that a problem with parameters l, j is reduced to two problems one with parameters $l-1, j$ and the other $l-1, j-1$. If the running time of the algorithm is $T(l, j)$, we get the recurrence $T(l, j) \leq T(l-1, j) + T(l-1, j-1) + O(1)$ which solves to $T(l, j)$ is in $O(l^j)$.

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Suppose as in the Theorem (2.1), P is a copolyhedron of affine dimension j_o in $\mathbf{R}^m$. Consider each of the (at most m^n) nonsingular $n \times n$ submatrices B of A. For each of these we can define an $n \times m$ matrix T by augmenting B^{-1} with 0 columns so that the possible corners of any K_b are of the form Tb for such T. For each such T, and each i, $1 \leq i \leq m$, consider the hyperplane $\{b : a^{(i)}Tb = b_i\}$ in $\mathbf{R}^m$. (Reminder : $a^{(i)}$ is the i th row of A.) There

are at most m^{n+1} such hyperplanes and so by lemma (2.3) , we have that P intersects at most $m^{(n+1)j_o}$ of the regions that these hyperplanes partition $\mathbf{R}^m$ into. It is not difficult to see that for fixed n, j_o, we can find these regions in polynomial time. For each such region U, there is a T_U such that $T_U b$ is in K_b for all $b \in U$; in other words $b - AT_U b$ is a nonnegative vector for all $b \in U$. Consider the m hyperplanes $(b - AT_U b)_i = M$ for $i = 1, 2, \ldots m$. By applying lemma (2.3) again, we see that U intersects at most m^{j_o} of the regions that these m hyperplanes partition space into. We partition U into these m^{j_o} or less parts. Thus we have found so far in polynomial time, a partition of P into copolyhedra $P_1, P_2, \ldots P_t$ with

$$t \leq m^{(n+2)j_o}$$

and associated with copolyhedron P_k in the above partition, we have a $T(P_k)$ and $I(P_k) \subseteq \{1, 2, \ldots m\}$ such that for all $b \in P_k$,

$$0 \leq (b - AT(P_k)b)_i \leq M \forall i \in I(P_k) \qquad \text{and}$$
$$(b - AT(P_k)b)_i > M \forall i \notin I(P_k).$$

For each $b \in P_k$, let $b' = b - AT(P_k)b$, let b'' be defined by $b''_i = b'_i$ for $i \in I(P_k)$ and $b''_i = M$ for other i. Let $b''' = b'' + AT(P_k)b$. Note that there is a linear transformation that maps each b to b''. Now by lemma (2.2), we see that for all $b \in P_k$, we have

$$K_{b'} + \mathbf{Z}^n = K_{b''} + \mathbf{Z}^n.$$

Note that b'' belongs to the copolytope

$$P' = \{b : b \in P; |b| \leq M\}$$

We apply Theorem (1.1) with this copolytope. I will show that an easy argument then gives us Theorem (2.1). To this end, let S_i be one of the sets in the partition of $P' \times \mathbf{R}^n$ that Theorem 1 yields. Corresponding to each such S_i we define one subset R_{ik} of $P_k \times \mathbf{R}^n$ for each k. Namely,

$$R_{ik} = \{(b, x) : b \in P_k \ ; \ (b'', x - T(P_k)b) \in S_i\}$$

It is easy to see that the R_{ik} have all properties 1,2, and 3 in the statement of Theorem (2.1) with a suitable choice of constant e. By Theorem (1.1), we have for all b in P_k,

$$(K_{b'} + \mathbf{Z}^n) \cap S_i(b'') \quad = \quad (K_{b''} + \mathbf{Z}^n) \cap S_i(b'') =$$
$$\left[\left\{ \bigcup_{B \in \mathcal{B}_i} ((K_{b''} + Z(B)) \cap F(B; T(B)b'')) \right\} + \mathbf{Z}^n \right] \cap S_i(b'').$$

Translating the sets on both sides of the above equation by $T(P_k)b$, we obtain

$$(K_b + \mathbf{Z}^n) \cap R_{ik}(b) =$$
$$\left[\left\{ \bigcup_{B \in \mathcal{B}_i} ((K_{b'''} + Z(B)) \cap F(B; T(B)b'' + T(P_k)b)) \right\} + \mathbf{Z}^n \right] \cap R_{ik}(b).$$

Since $K_{b'''} \subseteq K_b$, we may replace $K_{b'''}$ on the right hand side (rhs) of the above equation by K_b. (Note that then we would have lhs contained in the rhs. The converse is obvious.) Further, there is an affine transformation, say, Q that takes b to b''. So, we may now define the affine transformation corresponding to the basis B to be

$$T(B)Q + T(P_k)$$

to complete the proof.

3 Test sets for Integer Programs, $\forall\exists$ sentences and maximal lattice-free K_b

For linear programming problems, we know that if there is a feasible solution, there is a basic feasible one. This can be expressed as follows :

Suppose as before A is an $m \times n$ matrix. Consider as before, each of the (at most m^n) nonsingular $n \times n$ submatrices B of A. For each of these we can define an $n \times m$ matrix T by augmenting B^{-1} with 0 columns so that the

possible corners of any K_b are of the form Tb for such T. Then we can say that for all possible right hand sides b, if K_b is nonempty, then one of the Tb belongs to K_b. This section proves a similar theorem for Integer Programs.

Theorem (3.1) : Let A be an $m \times n$ matrix of integers of size ϕ with the property that $\{x : Ax \leq 0\} = \{0\}$ (or equivalently, K_b is bounded for all b). Let P be a copolyhedron in $\mathbf{R}^m$ of affine dimension j_o. For n, j_o fixed, there is a polynomial time algorithm that finds a partition of P into copolyhedra $P_1, P_2, \ldots P_r$ with $r \leq (mn\phi)^{j_o n^{d_n}}$ and for P_i, finds a set $\mathcal{T}_i$ of pairs (T, T') affine transformations where $T : \mathbf{R}^m \to \mathbf{R}^n$ and $T' : \mathbf{Z}^n \to \mathbf{Z}^n$ such that for all i and for all $b \in P_i$,

$$K_b \cap \mathbf{Z}^n \neq \emptyset \quad \Longleftrightarrow \quad \exists (T, T') \in \mathcal{T}_i \ : \ T' \lfloor Tb \rfloor \in K_b.$$

Further, for each i, the set $\mathcal{T}_i$ contains at most n^{4n} pairs (T, T').

Proof : Let $L = \mathbf{Z}^n$. Note that $K_b \cap L$ is empty iff $K_b + L$ does not contain 0. We apply Theorem (2.1) to get the partition of $P \times \mathbf{R}^n$ into $R_1, R_2, \ldots R_r$. Let $P_i = \{b : 0 \in R_i(b)\}$. It is easy to see that P_i is of the form $P_i'/\mathbf{Z}^l$ where P_i' is a copolyhedron and l is a constant (for fixed n). In fact a stronger statement is true. The P_i are actually copolyhedra. To see this, we have to go into the proof of Theorem (1.1). The partition of $P \times \mathbf{R}^n$ into $S_1, S_2, \ldots$ in that theorem (where $S_i = S_i'/\mathbf{Z}^l$) has the following property : for each $(b, x) \in S_i$ (with $b \in P, x \in \mathbf{Z}^n$), there is a unique $y \in \mathbf{Z}^l$ so that (b, x, y) belongs to S_i'. In fact, each component of y is of the form $F' \lfloor Fx \rfloor$ where F', F are affine transformations. This is easily proved by induction on n noting that in (4.5) of [8], the z is in fact forced to be $\lfloor \alpha + 1 - \beta \rfloor$.

Also, from Theorem (2.1), we have for b in P_i,

$$(K_b + \mathbf{Z}^n) \cap \mathbf{Z}^n =$$

$$\left[\left\{ \bigcup_{B \in \mathcal{B}_i} (K_b + Z(B)) \cap F(B; T(B)b) \right\} + \mathbf{Z}^n \right] \cap \mathbf{Z}^n.$$

The left hand side in the above equation is empty if and only if for each B, the unique lattice point $z_B(b)$ in $F(B; T(B)b)$ has the property that $z_B(b) - Z(B)$ does not intersect K_b. It is quite straightforward to see that for each

$p \in Z(B)$, we can find a pair of affine transformations (T, T') as required in the statement of Theorem (3.1), such that $z_B(b) - p = T'(\lfloor Tb \rfloor)$. This completes the proof of the theorem.

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I now give a decision procedure for deciding the truth or falsity of certain sentences in Presberger arithmetic.

Theorem (3.2) : There is an algorithm which takes as input an $m \times n$ matrix A and an $m \times p$ matrix B and an $m \times 1$ matrix b all made up of integers and a copolyhedron Q in $\mathbf{R}^{p+l}$ by a set of defining inequalities, decides whether the following sentence is true.

$$\forall y \in Q/\mathbf{Z}^l \quad \exists x \in \mathbf{Z}^n : \qquad Ax + By \leq b.$$

Further for fixed n, p, l, the algorithm runs in time bounded by a polynomial in the length of the input.

Remark : Note $\mathbf{R}^p$ and $\mathbf{Z}^p$ are both special cases of sets of the form $Q/\mathbf{Z}^l$. The first is obvious. For the second, we can make $l = p$ and $Q = \{(y, y) : y \in \mathbf{R}^p\}$.

Proof : Let $Q/\mathbf{R}^l = Q'$. The set Q' includes the set $Q/\mathbf{Z}^l$ - the set of all the y of interest. For y in Q', the quantity $b - By$ is in an affine subspace P of $\mathbf{R}^m$ of dimension p or less. So by theorem (3.1), we can find in polynomial time (since n, p are fixed) a partition of P into copolyhedra $P_1, P_2, \ldots P_r$ with

$$r \leq (n\phi m)^{pn^{dn}}$$

and for each P_i, a collection $\mathcal{T}_i$ of pairs of affine transformations (T, T') satisfying the conditions of that theorem. The sentence

$$\forall y \in Q/\mathbf{Z}^l \quad \exists x \in \mathbf{Z}^n : \qquad Ax + By \leq b$$

is false iff there is some P_i with the property that

$$\exists y \in P_i \cap Q/\mathbf{Z}^l : \forall (T, T') \in \mathcal{T}_i : T'\lfloor T(b - By) \rfloor \notin \{x : Ax \leq b - By\}.$$

This will be true iff one of the Mixed Integer programs set up below is feasible : Consider each of the $m^{n^{4n}}$ maps f from pairs (T, T') in $\mathcal{T}_i$ to $\{1, 2, \ldots m\}$. For each such map, we will have one MIP that asserts that there exists a $y \in P_i \cap Q/\mathbf{Z}^l$ with $(T' \lfloor T(b - By) \rfloor)$ violating the $f(T, T')$ th constraint for each (T, T'). Note that the floor of a real variable w can be expressed using a new integer variable which is constrained to be in the interval $(w - 1 \quad w]$. Also the condition that $y \in Q/\mathbf{Z}^l$ can be expressed by introducing l new integer variables. For convenience, order the pairs (T, T') and refer to them as $(T, T')_i$. The MIP will read as follows :

$$\exists y \in \mathbf{R}^p, z \in \mathbf{Z}^l, z_1, z_2, \ldots \in \mathbf{Z}^n : (y, z) \in Q; y \in P_i$$

$$T_i(b - By) - 1 < z_i \leq T_i(b - By); (AT'_i z_i)_{f(T,T')_i} > b_{f(T,T')_i}.$$

Clearly, we may solve each MIP for each P_i in turn and if one of them is feasible, return false for the sentence otherwise, true.

It is not difficult to see that the required bound on the running time.

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The rest of the section discusses properties of the set of right hand sides b for which $K_b \cap \mathbf{Z}^n$ is empty.

Let

$$LF(A, P) = \{b : b \in P, K_b \cap \mathbf{Z}^n = \emptyset\}.$$

Let $P_1, P_2, \ldots$ be the partition of P that Theorem (3.1) yields. Let

$$LF(A, P) = \cup_i LF(A, P_i).$$

$LF(A, P_i)$ can be described by linear constraints with the introduction of some extra integer variables as the following shows : we consider all mappings f of the following sort : f takes as argument a pair (T, T') in $\mathcal{T}_i$ and its range is $\{1, 2, \ldots m\}$. Let $V(i, f)$ be the set of b satisfying

- b belongs to P_i.
- $T'(\lfloor Tb \rfloor)$ violates constraint number $f(T, T')$ of the m constraints $Ax \leq b$.

To express the floor, we can introduce a new integer variable and linear constraint. Thus, we see that $LF(A, P_i)$ is the union of a polynomial number of sets each of the form copolyhedron/$\mathbf{Z}^l$ where l is a constant for fixed n, j_o. We use this discussion in a slightly different context below.

Suppose as above A is a fixed $m \times n$ matrix of integers with $\{x : Ax \leq 0\} = \{0\}$. For any $b \in \mathbf{R}^m$, as before, we let $K_b = \{x : x \in \mathbf{R}^n; Ax \leq b\}$. We say that a K_b is maximal-lattice-point free if it has no points of $\mathbf{Z}^n$ in its interior and any convex set that strictly contains K_b does. We can replace the last condition by the requirement that every facet of K_b have a lattice point interior to it [14]. By a theorem of Bell [1] and Scarf [16], a maximal lattice free K_b has at most 2^n facets. We choose all subsets of the m inequalities $Ax \leq b$ of cardinality at most 2^n, and for each subset, we will study the positions of the facets that result in maximal lattice free sets; we only incur an extra factor of m^{2^n} by this which is polynomially bounded for fixed n. Then arguing as for the case of lattice-point-free sets and adding the condition that each facet have a lattice point, we get the following theorem.

Theorem (3.3) : Suppose n is fixed. Then for any $m \times n$ integral matrix A, there exists a collection of sets $\{U_1, U_2, \ldots U_t\}$, where t is bounded by a polynomial in the size of A, and each U_i is of the form $U_i'/\mathbf{Z}^l$, where l is a constant, and U_i' is a copolyhedron such that the collection of maximal lattice point free sets K_b is precisely the collection$\{K_b : b \in U_1 \cup U_2 \cup \ldots U_t\}$.

Remark : Note that a similar theorem is not true for just lattice-point-free b - there we would have also needed to assume that m was fixed or else at least the affine dimension of P over which b varied was fixed. The theorem of Bell and Scarf helps us dispense with this assumption for *maximal* lattice point free sets.

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