Quick approximation to matrices and applications

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Abstract

We give algorithms to find the following simply described approximation to a given matrix. Given an $m \times n$ matrix $\mathbf{A}$ with entries between say -1 and 1, and an error parameter ϵ between 0 and 1, we find a matrix $\mathbf{D}$ (implicitly) which is the sum of $O(1/\epsilon^2)$ simple rank 1 matrices so that the sum of entries of any submatrix (among the 2^{m+n}) of $(\mathbf{A}-\mathbf{D})$ is at most ϵmn in absolute value. Our algorithm takes time dependent only on ϵ and the allowed probability of failure (not on m, n).

We draw on two lines of research to develop the algorithms: one is built around the fundamental Regularity Lemma of Szemerédi in Graph Theory and the constructive version of Alon, Duke, Leffman, Rödl and Yuster. The second one is from the papers of Arora, Karger and Karpinski, Fernandez de la Vega and most directly Goldwasser, Goldreich and Ron who develop approximation algorithms for a set of graph problems, typical of which is the maximum cut problem.

¿From our matrix approximation, the above graph algorithms and the Regularity Lemma and several other results follow in a simple way.

We generalize our approximations to multi-dimensional arrays and from that derive approximation algorithms for all dense Max-SNP problems.

1 Introduction

One motivation for this paper comes from certain graph problems, such as the **maximum weight cut problem**. Here we have a graph $G=(V,E)$ and weights $w: E \rightarrow \boldsymbol{R}$. For $S \subseteq V$ the cut $(S,\bar{S})$ is the set of edges with exactly one end in S. Its weight $w(S,\bar{S})$ is the total weight of its edges. The problem is to find a cut of maximum weight. It is easy to produce a cut (in polynomial time) which has at least $1/2$ of the weight of the maximum cut. Goemans and Williamson [17] made a breakthrough by devising a polynomial time algorithm which comes within a factor of .878 of optimal. This problem is Max-SNP hard; so from the PCP results of Arora, Lund, Motwani, Sudan and Szegedy [7] it is known that if we have a polynomial time approximation algorithm for every fixed factor less than 1 (or a Polynomial Time Approximation Scheme - PTAS) then NP would equal P.

However, Arora, Karger and Karpinski [6] gave the an algorithm for this problem which produces a cut of weight at least the maximum weight of a cut minus $\epsilon n^2 W$ where W is the maximum

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weight of an edge, and n is the number of vertices in the graph. This additive error bound implies a PTAS for the case when G has $\Omega(n^2)$ edges, each of weight 1 (henceforth referred to as the "dense case"). The running time of their algorithm is $O(n^{O(1/\epsilon^2)})$. Fernandez de la Vega [13] independently gave an $O(2^{1/\epsilon^{2+o(1)}} n^2)$ time algorithm for the unweighted Max Cut and Maximum Acyclic Subgraph problems with similar bounds. We give a *constant* time approximation scheme.

We describe a method of decomposing matrices into the sum of simple matrices plus an "error" matrix. Ignoring the error matrix makes many problems easy to solve. In this way we obtain algorithms which have running times $2^{\tilde{O}(1/\epsilon^2)}$ [1] in the probe model of computation (see Section 2.2). Our solutions are given *implicitly* and they can be quickly expanded to give explicit solutions.

Algorithms with comparable running times to ours for the above problems have been obtained earlier by Goldwasser, Goldreich and Ron through other means [18]. Sampling plays an important role in all of the above papers. Goldwasser, Goldreich and Ron showed that by appropriate sampling of a constant number of vertices one can determine with high probability whether a graph has a cut close to a certain weight and in addition provide some auxilliary information which implicitly defines the partition, enabling its quick construction later.

A second motivation for us comes from the Regularity Lemma of Szemerédi - a fundamental result in Graph Theory. This lemma gives a partition of the vertex set of any graph into a bounded number of pieces, so that the pieces satisfy some regularity properties – see Section 5.2 for a proper definition. While the original lemma was non-constructive, Alon, Duke, Leffman, Rödl and Yuster [1] gave a polynomial time algorithm to find the partition. In an earlier paper [15] we describe a related partition of the vertices with many fewer parts. This can be also be put to algorithmic use in solving maximum cut as well as several other problems, with an additive guarantee of error. In this context, we note that Duke, Lefmann and Rödl [11] used another decomposition, different from ours and Szemerédi's to approximate subgraph counts. The number of parts in their decomposition is closer to ours than Szemerédi's.

Using techniques from both these areas, we give here an algorithm for finding a natural approximation to matrices stated in the abstract.

This approximation (applied to the adjacency matrix of graphs) helps us solve (in a uniform way) the maximum cut and the other graph problems considered for example, in [18]. In addition, we solve a version of the **Quadratic Assignment Problem** [9, 24] which contains the **Minimum Linear Arrangement Problem** [16] as a special case.

We generalize our approximations to multi-dimensional matrices. Using this generalization, we give approximation algorithms with an additive error guarantee for all **Max-SNP** problems. (The class Max-SNP was introduced by Papadimitriou and Yanakakis [23]. We will briefly explain the class in Section 7.)

Perhaps a central point of our paper is that all our algorithms are obtained from the matrix approximation theorem with minimum effort.

We note that there has been fair bit of success in designing polynomial time approximation schemes for certain graph problems (such as the Max Cut problem) on *dense graphs*, as mentioned above and other problems like the QAP (Arora, Frieze and Kaplan [5]), the existence of such schemes for general graphs would imply that NP=P by the powerful results of Arora, Lund, Motwani, Sudan and Szegedy [7]. This mirrors the situation in approximate counting where dense problems have sometimes been easier to attack – Annan [4], Broder [8], Jerrum and Sinclair [21], Dyer, Frieze and Jerrum [12] and Alon, Frieze and Welsh [2]. We have not as yet

[1] The $\tilde{O}$ notation hides polynomial factors in $\log 1/\epsilon$, $\log 1/\delta$. δ will be our probability of error.

found a way of using our decomposition for such approximate counting problems.

We also use our matrix approximation Theorem (applying it again to adjacency matrices of graphs) to derive a constructive version of Szemerédi's Regularity Lemma. In Section 5.1 we show how to use the matrix approximation Theorem, another partition where the number of parts in the partition grows more slowly than Szemerédi's. In our partition the logarithm of the number of parts is polynomial in $1/\epsilon$ whereas in Szemerédi's only $\log^*$ of the number of parts is – necessarily so, Gowers [19]. Of course our partition does not have as strong a regularity property as Szemerédi's. However, the weaker conclusion is enough for certain purposes; in fact, as we mentioned earlier, we may also derive our algorithms from this version, see [15].

2 Statement of Results

2.1 Notation

We will be mainly concerned here with matrices having rows indexed by a set R and columns indexed by a set C, $|R| = m$ and $|C| = n$. The ith row of $\mathbf{M}$ is denoted by $\mathbf{M}_i$. We use the notation that for any vector $x \in \mathbf{R}^n$, and any subset S of coordinates, $x(S) = \sum_{i \in S} x_i$. For such an $R \times C$ matrix $\mathbf{M}$ we use several norms:

$$
\begin{aligned}
||\mathbf{M}||_\infty &= \max_{(i,j)\in R\times C} |\mathbf{M}(i,j)|. \\
||\mathbf{M}||_F &= \sqrt{\sum_{(i,j)\in R\times C} \mathbf{M}(i,j)^2}. && \text{Frobenius Norm} \\
||\mathbf{M}||_C &= \max_{S\subseteq R, T\subseteq C} |\mathbf{M}(S,T)|, && \text{Cut Norm}
\end{aligned}
$$

where

$$\mathbf{M}(S,T) = \sum_{(i,j)\in S\times T} \mathbf{M}(i,j).$$

We note that

$$||\mathbf{M}||_C \leq \sup_{x \in \boldsymbol{R}^n\setminus\{0\}} \frac{||\mathbf{M}x||_1}{||x||_\infty} \leq 4||\mathbf{M}||_C. \tag{1}$$

The lower bound follows from considering $x \in \{0,1\}^n$. For the upper bound observe that maximum of (the convex function) $||\mathbf{M}x||_1$ over (the polytope) $||x||_\infty \leq 1$ occurs at (an extreme point) $x \in \{-1,1\}^n$.

Given $S \subseteq R$, $T \subseteq C$ and real value d we define the $R \times C$ *Cut Matrix* $\mathbf{C} = CUT(S,T,d)$ by

$$\mathbf{C}(i,j) = \begin{cases} d & \text{if } (i,j) \in S \times T, \\ 0 & \text{otherwise.} \end{cases}$$

Note that a cut matrix has rank one.

2.2 Model of Computation

We will design algorithms that run in *constant time.* Since the data size for these problems is unbounded, we must be precise about what we mean. We use the *Probe* model in which we assume that given $(i, j) \in R \times C$ and matrix $\mathbf{A}$ we can in O(1) time determine $\mathbf{A}(i, j)$, by a "probe". Our results state that many problems can be *implicitly* solved using a constant number of random probes. By implicitly, we mean that we obtain a short description of a solution, which can be "expanded" explicitly in polynomial time, usually $O(m+n)$ time. Our results will mostly be stated in this model, which was introduced in [18].

2.3 Matrix Decompositions

A *Cut Decomposition* expresses a matrix $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \cdots + \mathbf{D}^{(s)} + \mathbf{W}. \tag{2}$$

Here $\mathbf{D}^{(t)} = CUT(R_t, C_t, d_t)$ for $t = 1, 2, \ldots, s$. Such a decomposition has *width* s, *coefficient length* $(d_1^2 + \cdots + d_s^2)^{1/2}$ and *error* $||\mathbf{W}||_C$.

Theorem 1 *Suppose $\mathbf{A}$ is an $R \times C$ matrix and suppose ϵ, δ are reals in the interval $(0,1)$. Then in time $\tilde{O}(\epsilon^{-12}\delta^{-1})$, we can, with probability $1-\delta$, find a cut decomposition of width $O(\epsilon^{-4})$, coefficient length at most $\sqrt{27}||\mathbf{A}||_\infty$ and error at most $\epsilon mn||\mathbf{A}||_\infty$.*

The next theorem claims a decomposition of smaller width. It takes longer to produce. In this algorithm we can avoid the dependence on $||\mathbf{A}||_\infty$.

Theorem 2 *Let $\mathbf{A}, \epsilon, \delta$ be as in Theorem 1. Then in time $2^{\tilde{O}(1/\epsilon^2)}/\delta^2$, we can, with probability $1-\delta$, find a cut decomposition of width $O(\epsilon^{-2})$, coefficient length at most $\sqrt{27}||\mathbf{A}||_F/\sqrt{mn}$ and error at most $\epsilon\sqrt{mn}||\mathbf{A}||_F$.*

If $\mathbf{A}$ is a symmetric matrix then it could be useful to have what we call a *symmetric decomposition*. This is easily done. If $\mathbf{A} = \mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \cdots + \mathbf{D}^{(s)} + \mathbf{W}$ then we use the decomposition

$$\begin{aligned} \mathbf{A} &= \frac{\mathbf{D}^{(1)}}{2} + \frac{(\mathbf{D}^{(1)})^T}{2} + \cdots + \frac{\mathbf{D}^{(s)}}{2} + \frac{(\mathbf{D}^{(s)})^T}{2} + \frac{\mathbf{W}+\mathbf{W}^T}{2}. \\ &= \bar{\mathbf{D}}^{(1)} + \bar{\mathbf{D}}^{(2)} + \cdots + \bar{\mathbf{D}}^{(2s-1)} + \bar{\mathbf{D}}^{(2s)} + \bar{\mathbf{W}}. \end{aligned} \tag{3}$$

The $\bar{\mathbf{D}}^{(i)}$ are not necessarily symmetric, but we use them in pairs indexed by $2j-1$ and $2j$. Note that

$$||\bar{\mathbf{W}}||_C \leq ||\mathbf{W}||_C.$$

Theorems 1 and 2 are proved in Sections 4.2 and 4.3.

2.4 Approximation Algorithms

Theorem 3 Max-Cut *Let G denote the complete graph with vertex set V and edge weights $w : V \times V \to [-1, 1]$. Then in time $2^{\tilde{O}(1/\epsilon^2)} \log 1/\delta$ we can find a cut $(S^*, V \setminus S^*)$ such that with probability at least $1-\delta$,*

$$w(S^*, V \setminus S^*) \geq w(S, V \setminus S) - \epsilon n^2$$

for all $S \subseteq V$.

(If the edge weights are in $[-W, W]$ then by scaling we see that the error is at most $\epsilon W n^2$.)

We can prove a related theorem on the *conductance* of graphs. Suppose $G(V,E)$ is an undirected graph. Let us define the conductance of G denoted $\mathrm{Cond}(G)$ by

$$\mathrm{Cond}(G) = \min_{S \subseteq V} \mathrm{Cond}(S)$$

where

$$\mathrm{Cond}_S = \frac{|\{(u,v) : u \in S; v \in V \setminus S\}|}{|S||V \setminus S|}.$$

When the degrees of all vertices are equal, this coincides with the definition of Jerrum and Sinclair [21].

Theorem 4 *Given a graph G and $\epsilon, \delta > 0$, there is a $O(n2^{\tilde{O}(1/\epsilon^2)} \log 1/\delta)$ time algorithm which returns a real number τ so that with probability at lesst $1-\delta$, we have*

$$|\mathrm{Cond}(G) - \tau| \leq \epsilon.$$

We now consider the QAP. We focus on the Koopmans-Beckmann version of the QAP. Here one is given a set of n items V which have to be assigned to a set of n locations X, one per location. We are given two $n \times n$ non-negative matrices $\mathbf{T}, \mathbf{D}$. Here $\mathbf{T}(i,i')$ is the amount of *traffic* between item i and i' and $\mathbf{D}(x,x')$ is the *distance* between location x and x'. If item i is assigned to location $\pi(i)$ for $i \in [n]$ the total cost $c(\pi)$ is defined by

$$c(\pi) = \sum_{i=1}^{n} \sum_{i'=1}^{n} \mathbf{T}(i,i')\mathbf{D}(\pi(i), \pi(i')). \tag{4}$$

The problem is to minimise $c(\pi)$ over all bijections $\pi : V \to X$.

A typical example is where a location is a room in a building (e.g. hospital) and each item is a facility of some sort (e.g. operating theatre, intensive care unit etc.) and the total cost is the sum over pairs of facilities of the product of traffic intensity and distance.

We will restrict our attention to the case where the n locations are the points of a finite metric space X with metric $\mathbf{D}$. We assume that

1. diam(X)=1 i.e. $\max_{x,y} \mathbf{D}(x,y) = 1$. (This can be assumed w.l.o.g. by scaling).
2. For all $\epsilon > 0$ there exists a partition $X = X_1 \cup X_2 \cup \cdots \cup X_\ell$, $\ell = \ell(\epsilon)$, such that $\mathrm{diam}(X_j) \leq \epsilon$, for $1 \leq j \leq \ell$. We call this an $\epsilon - refinement$ of X.

   Furthermore this partition is computable in time polynomial in n and $1/\epsilon$ – for the cases we have in mind, this will be insignificant compared with that required by the rest of the algorithm.

We call this the *metric* QAP.

The **Minimum Linear Arrangement** problem [16] where $X = \{0, 1/n, 2/n, \dots, 1\}$ is a special case. Partition X is just $\lceil 1/\epsilon \rceil$ intervals of length roughly equal to ϵ, each containing roughly ϵn points.

Similiarly, if the points are in $[0,1]^d$ then we divide this into $\lceil 1/\epsilon \rceil^d$ subcubes in the natural way. Here $\mathrm{diam}(X) \leq d^{1/2}$ and we need to scale to get the precise formulation.

We will also assume that $\mathbf{T}(i,i') \leq 1$ for all i, i' and this can be achieved by scaling. Let π^* denote the permutation which minimises c.

Theorem 5 *There is a randomised algorithm algorithm for the metric QAP which, with probability at least $1-\delta$, produces a permutation π_ϵ such that $c(\pi_\epsilon) \le c(\pi^*) + \epsilon n^2$ and which runs in expected time $\phi(\epsilon)\log 1/\delta$ for some function ϕ.*

We next look at the related **Maximum Acyclic Subgraph Problem**. Here we are given a (weighted) digraph D with adjacency matrix $\mathbf{T}$ and the problem is to find the maximum (weight) subset of the edges which induces an acyclic digraph.

Theorem 6 *Let D be an edge weighted digraph with arc (i,j) having weight $\mathbf{T}(i,j)$ where $||\mathbf{T}||_\infty \le 1$. There is a randomised algorithm which in time $2^{\tilde{O}(1/\epsilon^2)}\log 1/\delta$, can with probability at least $1-\delta$, find an edge set $\tilde{E} \subseteq E$ which induces an acyclic subgraph and*

$$\mathbf{T}(\tilde{E}) \ge \mathbf{T}(E^*) - \epsilon n^2,$$

where E^* is the optimal solution.

Theorems 3 – 6 are proved in Section 3.

2.5 Graph Partitions

The research that led to this paper was sparked by our realisation that given a decomposition promised by Szemerédi's Regularity Lemma, we could easily get a good approximation to Max-Cut (Theorem 3). We then realised that we did not really need such a fine partition and that a partition adequate for Theorem 3 can be computed more easily.

In Section 5 we describe Szemerédi's partition as well as our weaker Pseudo-Regular partition. We then show how to use our matrix decomposition algorithms to find such partitions.

2.6 Higher Dimensional Matrices

In Section 6 we will consider higher dimensional matrices. We will extend our notion of a cut decomposition. We will prove (Theorem 11) that a recursive application of our 2-dimensional algorithms can be used to yield good decompositions in higher dimensions.

This will lead us naturally to consider hypergraph versions of the partitions discussed in Section 5. We will show how to construct regular partitions of hypergraphs.

Finally, using our higher dimensional matrix decompositions we will show in Section 7 how to obtain a PTAS for a dense instance of any optimisation problem in MAX-SNP.

3 Combinatorial Problems

3.1 Max-Cut

In this section we prove Theorem 3 and show how to use the matrix approximation to find approximately the maximum weight cut in a graph $G(V,E)$. This illustrates the method used for all problems. In Section 7, we use the same method generalized to multi-dimensional matrices to solve approximately any general Max-SNP problem; but it is easier to understand the method in the simple setting of Max-Cut in graphs first.

We take the matrix $\mathbf{A}$ with $\mathbf{A}(i,j)$ equal to the weight $w(i,j)$ of the edge (i,j). We use Theorem 2 to implicitly find cut matrices $\mathbf{D}^{(1)}, \mathbf{D}^{(2)}, \ldots \mathbf{D}^{(s)}$, $s = O(1/\epsilon^2)$ with $\mathbf{D}^{(t)} = Cut(R_t, C_t, d_t)$ such that with probability at least 7/8

$$||\mathbf{A} - (\mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \cdots + \mathbf{D}^{(s)})||_C \leq \epsilon n ||\mathbf{A}||_F/10 \leq \epsilon n^2/10. \tag{5}$$

This takes time $2^{\tilde{O}(1/\epsilon^2)}$.

Suppose $(S, \bar{S})$ is a cut in the graph. Then, $\mathbf{A}(S, \bar{S})$ is the weight of this cut and (5) implies

$$|\mathbf{A}(S, \bar{S}) - (\mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \cdots + \mathbf{D}^{(s)})(S, \bar{S})| \leq \epsilon n^2/10. \tag{6}$$

But, $\mathbf{D}^{(t)}(S, \bar{S}) = d_t |S \cap R_t| |\bar{S} \cap C_t|$ and so

$$\sum_{t=1}^{s} \mathbf{D}^{(t)}(S, \bar{S}) = \sum_{t=1}^{s} d_t f_t g_t \tag{7}$$

where

$$f_t = |S \cap R_t| \text{ and } g_t = |\bar{S} \cap C_t| \qquad \text{for } t = 1, 2, \ldots s. \tag{8}$$

We let $\nu = \lfloor \epsilon n/(9\sqrt{27}) \rfloor$, (see our bound on the coefficient length of the decomposition), and we consider approximations to f_t, g_t defined by

$$\bar{f}_t = \left\lfloor \frac{f_t}{\nu} \right\rfloor \nu, \qquad \bar{g}_t = \left\lfloor \frac{g_t}{\nu} \right\rfloor \nu. \tag{9}$$

We see using the fact that $|d_t| \leq \sqrt{27}$,

$$\sum_{t=1}^{s} |f_t g_t d_t - \bar{f}_t \bar{g}_t d_t| \leq \sqrt{27} s (2\nu n + \nu^2) \leq 3\sqrt{27} \nu n s. \tag{10}$$

We see from (6), (7), (8) and (10) that the values $\bar{f}_t, \bar{g}_t$, $1 \leq t \leq s$ almost determine the weight of the corresponding cut. Thus our problem is reduced to finding the best values for $\bar{f}, \bar{g}$ and a corresponding cut. Now each $\bar{f}_t$ and $\bar{g}_t$ take on one of only $O(1/\epsilon^3)$ values and so we can afford to enumerate all $O(1/\epsilon^3)^{2s}$ possible values of $\bar{f}, \bar{g}$, try to find a cut for each and take the best cut found. Finding a cut for a given set of values $\bar{f}, \bar{g}$ is an integer program which we replace by its linear relaxation.

Max-Cut Algorithm

Let $\mathcal{P}$ be the coarsest partition of V (with at most 2^{2s} parts in it) such that each R_t, C_t is the union of sets in $\mathcal{P}$. We explicitly construct a representation of $\mathcal{P}$. We (i) construct a decision tree which has a leaf for each $P \in \mathcal{P}$ such that for any $v \in V$ we can determine P containing v in $O(1/\epsilon^2)$ time and (ii) for each R_t, C_t we make a list of those P which are its subsets.

Let $K = \{0, 1, 2, \ldots, \lceil 10\sqrt{27} s/\epsilon \rceil\}$. For each $(\bar{f}, \bar{g}) \in \nu K^{2s}$ we define the following integer program: for each $P \in \mathcal{P}$, x_P represents the unknown $|S \cap P|$.
$IP_{\bar{f},\bar{g}}$: find an integer solution to

$$\begin{array}{lllll} 0 & \leq & x_P & \leq & |P| & \forall\, P \in \mathcal{P} \\ \bar{f}_t & \leq & \displaystyle\sum_{P \subseteq R_t} x_P & < & \bar{f}_t + \nu & 1 \leq t \leq s \\ \bar{g}_t & \leq & \displaystyle\sum_{P \subseteq C_t} (|P| - x_P) & \leq & \bar{g}_t + \nu. & \end{array}$$

Let $LP_{\bar{f},\bar{g}}$ denote the linear relaxation of this problem. This program is feasible whenever $\bar{f},\bar{g}$ are derived from a set S as in (8) and (9) – take $x_P = |S \cap P|$, in which case $\sum_{P\subseteq R_t} x_P = |S \cap R_t|$ etc..

If $LP_{\bar{f},\bar{g}}$ is feasible we round down each x_P to the nearest integer (below it) to get y_P. Then, we have for each t, the upper bound on $\sum_{P\subseteq R_t} x_P$ is still satisfied i.e $\sum_{P\subseteq R_t} y_P < \bar{f}_t + \nu$. Also we have

$$\sum_{P\subseteq R_t} y_P \geq \bar{f}_t - 2^{2s}.$$

Similary we have

$$\bar{g}_t \leq \sum_{P\subseteq C_t} (|P| - y_P) \leq \bar{g}_t + \nu + 2^{2s}.$$

After finding the y_P, we take any $S^* = S^*(\bar{f},\bar{g})$ with $|S^* \cap P| = y_P$ for all $P \in \mathcal{P}$ (such S^*'s exist as $\mathcal{P}$ is a partition). We have

$$\begin{aligned} ||R_t \cap S^*| - \bar{f}_t| &\leq \nu + 2^{2s} \leq 2\nu \\ ||C_t \cap \bar{S}^*| - \bar{g}_t| &\leq \nu + 2^{2s} \leq 2\nu, \end{aligned}$$

for n high enough, since $2^{2s} \in 2^{\tilde{O}(1/\epsilon^2)}$.

This implies that (arguing as in (10)), for each feasible set of $\bar{f}_t, \bar{g}_t$, we can find S^* with

$$\Big|\sum_{t=1}^{s} |R_t \cap S^*||C_t \cap \bar{S}^*| d_t - \sum_{t=1}^{s} \bar{f}_t \bar{g}_t d_t\Big| \leq 5\sqrt{27}\nu n s. \tag{11}$$

So, taking the best $S^*(\bar{f},\bar{g})$ as $(\bar{f},\bar{g})$ runs over νK^{2s}, we see from (6), (10) and (11) that we get a cut which is at least the maximum minus $8\sqrt{27}\nu n s + \epsilon n^2/10 \leq \epsilon n^2$ as claimed. We manage this with probability at least 7/8. By repeating $O(\log 1/\delta)$ times and taking the best cut found we obtain our theorem. □

All subsequent algorithms use this strategy. So when we say "compute a decomposition satisfying ..." we implicitly mean compute one with probability at least 7/8. Repetition of the decomposition plus optimisation $O(\log 1/\delta)$ times is used to improve the probability to $1 - \delta$. We will not always say this explicitly in what follows.

3.2 Conductance

In this section we prove Theorem 4. Let $\epsilon_1 = \epsilon/8$ and assume ϵ is sufficiently small. Let $d(v)$ denote the degree of vertex v and for $S \subseteq V$ let $d(S) = \sum_{v\in S} d(v)$. We first estimate all the degrees. For this we pick an independent sequence $v_1, v_2, \ldots, v_T$ of randomly chosen vertices, $T = 64 \log n/\epsilon_1^2$. Let $d'(v) = nB_v/T$ where B_v is the number of i such that v_i is adjacent to v. B_v has distribution $Bin(T, d(v)/n)$ and so (see Corollary A.7 of Alon and Spencer [3])

$$\mathbf{Pr}(|d'(v) - d(v)| \geq \epsilon_1 n/8) = \mathbf{Pr}(|B_v - Td(v)/n| \geq T\epsilon_1/8) \leq n^{-2}.$$

So assume that

$$|d'(v) - d(v)| \leq \epsilon_1 n/8 \qquad \forall v \in V.$$

Case 1: There is some $v \in V$ with $d'(v) \leq \epsilon_1 n$. Then, $d(v) \leq \epsilon n/4$ and so with $S = \{v\}$, we get

$$\frac{|\{(u,v) : u \in S; v \in V \setminus S\}|}{|S||V \setminus S|} \leq \epsilon/3,$$

whence we may output $\tau = 0$ and stop.

Case 2 $d'(v) \geq \epsilon_1 n$, $\forall v \in V$. Now for any $S \subseteq V$ with $|S| \leq \epsilon_1^2 n/2$, we have

$$(1-\epsilon_1)\frac{d(S)}{n|S|} \leq \frac{d(S)-|S|^2}{|S||V\setminus S|} \leq \mathrm{Cond}_S = \frac{|\{(u,v) : u \in S; v \in V \setminus S\}}{|S||V\setminus S|} \leq \frac{d(S)}{n|S|}(1+\epsilon_1).$$

We then have that,

$$(1-\epsilon_1)\frac{d(S)}{n|S|} \leq \mathrm{Cond}_S \leq (1+\epsilon_1)\frac{d(S)}{n|S|}, \qquad \forall |S| \leq \epsilon_1^2 n/2.$$

So, $\min_{S:|S|\leq\epsilon_1^2 n/2} \mathrm{Cond}_S$ can be found with error at most ϵ by just arranging the vertices in increasing order of (estimated) degrees and examining the vertices in this order. (We will not give the simple details.)

Now we deal with sets S with $\epsilon_1^2 n/2 \leq |S| \leq n/2$. For this, we start by finding an approximation $\mathbf{D}$ to $\mathbf{A}$ so that for all $S \subseteq V$, we have

$$|\mathbf{A}(S,\bar{S}) - \mathbf{D}(S,\bar{S})| \leq \epsilon\epsilon_1^2 n^2/3. \tag{12}$$

For any $S \subseteq V$, let

$$\mathrm{Cond}_S^* = \frac{\mathbf{D}(S, V\setminus S)}{|S|\,|V\setminus S|}$$

Then for these large S, we have, from (12),

$$|\mathrm{Cond}_S - \mathrm{Cond}_S^*| \leq \epsilon,$$

so it suffices to find the minimum of Cond_S^* . This is done in a manner similar to the maximum cut problem in time $2^{\tilde{O}(1/\epsilon^6)}$.

3.3 Quadratic Assignment

In this section we prove Theorem 5. Our assumption about the metric allows us to define, for any $\epsilon > 0$, an $\ell(\epsilon) \times \ell(\epsilon)$ matrix $\hat{\mathbf{D}}_\epsilon$ such that if $x \in X_j$ and $x' \in X_{j'}$ then $|\mathbf{D}(x,x') - \hat{\mathbf{D}}(j,j')| \leq 2\epsilon$.

We start by applying Theorem 2 and decomposing

$$\mathbf{T} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)} + \cdots + \mathbf{T}^{(s)} + \mathbf{W}$$

as a sum of cut matrices $\mathbf{T}^{(i)} = CUT(R_i, C_i, d_i)$ and $||\mathbf{W}||_C \leq \epsilon_0 n^2$, $\epsilon_0 = \epsilon/(8\ell_0^2)$ where $\ell_0 = \ell(\epsilon/16)$. Thus for a bijection $\pi : V \to X$ we have

$$c(\pi) = \sum_{k=1}^{s}\sum_{i,j=1}^{n} \mathbf{T}^{(k)}(i,j)\mathbf{D}(\pi(i),\pi(j)) + \Delta_1 \tag{13}$$

where

$$\begin{aligned}\Delta_1 &= \sum_{i,j=1}^{n} \mathbf{D}(\pi(i),\pi(j))\mathbf{W}(i,j)\\ &= \sum_{p,q=1}^{\ell_0} \sum_{\substack{\pi(i)\in X_p\\ \pi(j)\in X_q}} (\hat{\mathbf{D}}_{\epsilon/8}(p,q) + \mathrm{err}_{i,j})\mathbf{W}(i,j),\end{aligned}$$

where $|\mathrm{err}_{i,j}| \leq \epsilon/8$.

Now note that $\left|\sum_{i,j=1}^n \mathrm{err}_{i,j} \mathbf{W}(i,j)\right| \leq \epsilon n^2/8$ and

$$\begin{aligned}
\left| \sum_{p,q=1}^{\ell_0} \sum_{\substack{\pi(i)\in X_p \\ \pi(j)\in X_q}} \hat{\mathbf{D}}_{\epsilon/8}(p,q)\mathbf{W}(i,j) \right| &\leq \sum_{p,q=1}^{\ell_0} |\mathbf{W}(\pi^{-1}(X_p), \pi^{-1}(X_q))| \\
&\leq \ell_0^2 ||\mathbf{W}||_C \\
&\leq \frac{\epsilon n^2}{8}.
\end{aligned}$$

Thus $\Delta_1 \leq \epsilon n^2/4$.

We compute an ϵ_1-refinement of X, $\epsilon_1 = \epsilon/(8\sqrt{\gamma}s)$ (A somewhat similar idea was used in ([5])). Then let $S_i^{(\pi)} = \pi^{-1}(X_i)$ for $1 \leq i \leq \ell = \ell(\epsilon_1)$. In which case we can write

$$\sum_{k=1}^{s} \sum_{i,j=1}^{n} \mathbf{T}^{(k)}(i,j)\mathbf{D}(\pi(i),\pi(j)) = \sum_{k=1}^{s} \sum_{i,j=1}^{\ell} d_k |R_k \cap S_i^{(\pi)}|\,|C_k \cap S_j^{(\pi)}| \hat{\mathbf{D}}_{\epsilon_1}(i,j) + \Delta_2 \tag{14}$$

where $|\Delta_2| \leq 2\sqrt{27}s\epsilon_1 n^2 \leq \epsilon n^2/4$. We use the fact that $|d_k| \leq \sqrt{27}$, the bound on the coefficient length of the decomposition.

We let $\nu = \lfloor \epsilon n/(12\sqrt{27}\ell s) \rfloor$ and

$$x_{i,k}^{\pi} = \lfloor |R_k \cap S_i^{(\pi)}|/\nu \rfloor \text{ and } y_{j,k}^{\pi} = \lfloor |C_k \cap S_j^{(\pi)}|/\nu \rfloor \qquad \forall\, i,k.$$

So,

$$\sum_{k=1}^{s} \sum_{i,j=1}^{\ell} d_k |R_k \cap S_i^{(\pi)}|\,|C_k \cap S_j^{(\pi)}| \hat{\mathbf{D}}_{\epsilon_1}(i,j) = \nu \sum_{k=1}^{s} \sum_{i,j=1}^{\ell} d_k x_{i,k}^{\pi} y_{j,k}^{\pi} + \Delta_3 \tag{15}$$

where $|\Delta_3| \leq 3\sqrt{27}\nu\ell s n \leq \epsilon n^2/4$.

As we vary π each of $x_{i,k}^{\pi}, y_{j,k}^{\pi}$ takes on one of $O(\ell/\epsilon^3)$ values. So, as in the case of max-cut we try all $O((\ell/\epsilon^3)^{2\ell s})$ choices for the vector $(x_{i,k}^{\pi}, y_{j,k}^{\pi})$ and take the best "feasible" one.

In a similar manner to that in Section 3.1 we implicitly compute the coarsest partition $\mathcal{P}$, $|\mathcal{P}| \leq 4^s$ such that each R_k and C_k is the union of members of $\mathcal{P}$. We introduce variables $\lambda_{i,P}$, $P \in \mathcal{P}, 1 \leq i \leq \ell$ and for each trial vector $(\xi_{i,k}, \eta_{j,k})$ we check the feasibility of the linear system

$$\begin{aligned}
\xi_{i,k} \quad &\leq \sum_{P\subseteq R_k} \lambda_{i,P} \leq \quad \xi_{i,k} + 1 \qquad \forall i,k \\
\eta_{i,k} \quad &\leq \sum_{P\subseteq C_k} \lambda_{i,P} \leq \quad \eta_{j,k} + 1 \qquad \forall j,k \\
\lfloor |X_i|/\nu \rfloor \quad &\leq \sum_{i=1}^{\ell} \lambda_{i,P} \leq \quad \lfloor |X_i|/\nu \rfloor + 1
\end{aligned} \tag{16}$$

Assuming feasibility in (16) we round down a solution $\lambda_{i,P}$ to integer values $\mu_{i,P}$ and choose any bijection which maps between $\mu_{i,P}\nu$ and $(\mu_{i,P}+1)\nu$ members of P to X_i for every i, P. In this way, we find a solution π_ϵ such that for any other solution π, $c(\pi_\epsilon) \leq c(\pi) + 3n\nu + \Delta_1 + \Delta_2 + \Delta_3 \leq c(\pi) + \epsilon n^2$ and this proves Theorem 5.

3.4 Maximum Acyclic Subgraph

In this section we prove Theorem 6. Let $\nu = \lfloor \epsilon n/2 \rfloor$ and $X = [n]$. Let $X_1, X_2, \ldots, X_\ell$ be a partition of X into sets of size ν or $\nu + 1$ such that if $i < j$ then $\max X_i < \min X_j$. We re-formulate the problem as one of finding a bijection $\pi : V \to X$ which maximises

$$c(\pi) = \sum_{\substack{(x,y)\in E \\ \pi(x)<\pi(y)}} \mathbf{T}(x,y).$$

We define a distance matrix $\mathbf{D}$ by

$$\mathbf{D}(x,y) = \begin{cases} 1 & x \in X_i, y \in X_j, i < j, \\ 0 & \text{otherwise.} \end{cases}$$

We then observe that

$$c(\pi) = \sum_{i=1}^{n} \sum_{i'=1}^{n} \mathbf{T}(i,i')\mathbf{D}(\pi(i),\pi(i')) + \Delta,$$

where $|\Delta| \leq \epsilon n^2/2$. Comparing with (4) we see that we can proceed as in the previous section. The reader might be troubled by the fact that $\mathbf{D}$ does not define a metric. However this is not essential. All we need to be able to do is define a good approximating matrix $\hat{\mathbf{D}}$ and here $\hat{\mathbf{D}}(i,j) = 1_{i<j}$ will suffice.

4 Computing Decompositions

4.1 Existence

In order to help motivate the more technical constructive proofs, we first give a simple non-constructive version of our decomposition theorems.

Theorem 7 *Suppose* $\mathbf{A}$ *is a real* $m \times n$ *matrix with rows* R *and columns* C. *Then there exist cut matrices* $\mathbf{D}^{(1)}, \mathbf{D}^{(2)}, \ldots \mathbf{D}^{(s)}$, $\mathbf{D}^{(t)} = CUT(R_t, C_t, d_t)$ *for* $1 \leq t \leq s \leq 1/\epsilon^2$ *such that if*

$$\mathbf{W}^{(t)} = \mathbf{A} - (\mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \cdots + \mathbf{D}^{(t)})$$

then

$$|\mathbf{W}^{(s)}(S,T)| \leq \epsilon\sqrt{|S|\,|T|}||\mathbf{A}||_F \forall S \subseteq R, T \subseteq C. \tag{17}$$

$\forall S \subseteq R, T \subseteq C.$

Proof Assume inductively that we have found $t < 1/\epsilon^2$ cut matrices

$$\mathbf{D}^{(j)} = CUT(R_j, C_j, d_j), \quad 0 \leq j \leq t \quad (\mathbf{D}^{(0)} = 0),$$

such that $\mathbf{W} = \mathbf{W}^{(t)}$ satisfies

$$||\mathbf{W}||_F^2 \leq (1 - \epsilon^2 t)||A||_F^2.$$

We show that either (17) holds, proving the theorem (with $s = t$) or else we can find a decomposition with $t+1$ matrices that also satisfies (17). By inspection, (17) precludes $t > 1/\epsilon^2$ and the theorem is proved. So assume that there exis $R, T \subseteq C$ such that $|\mathbf{W}(S,T)| \geq \epsilon\sqrt{|S|\,|T|}||A||_F$. Let $R_{t+1} = S, C_{t+1} = T$ and $d_{t+1} = \mathbf{W}(S,T)/(|S||T|)$. Then

$$\begin{aligned}
||\mathbf{W}^{(t+1)}||_F^2 - ||\mathbf{W}||_F^2 &= ||\mathbf{W} - \mathbf{D}^{(t+1)}||_F^2 - ||\mathbf{W}||_F^2 \\
&= \sum_{i \in R_{t+1}, j \in C_{t+1}} ((\mathbf{W}(i,j) - d_{t+1})^2 - \mathbf{W}(i,j)^2) \\
&= -|R_{t+1}||C_{t+1}|d_{t+1}^2 \\
&= -\frac{\mathbf{W}(R_{t+1}, C_{t+1})^2}{|R_{t+1}|\,|C_{t+1}|} \\
&\leq -\epsilon^2||\mathbf{A}||_F^2.
\end{aligned}$$

The theorem follows. $\square$

Remark: For $m = 1$ the theorem says that we can write an arbitrary real n-vector a as the weighted sum of $\leq \epsilon^{-2}$ 0-1 vectors and an error vector w for which $|w(S)| \leq \epsilon\sqrt{n}|a|$ for all $S \subseteq [n]$.

4.2 First Algorithm

Proof of Theorem 1. We assume that $||\mathbf{A}||_\infty = 1$. The general case is dealt with by scaling.

At a general stage, for some $t \geq 0$, we will, with sufficently high probability, have found cut matrices $\mathbf{D}^{(1)}, \ldots \mathbf{D}^{(t)}$ such that $\mathbf{W}^{(t)} = \mathbf{A} - (\mathbf{D}^{(1)} + \cdots + \mathbf{D}^{(t)})$ satisfies

$$||\mathbf{W}^{(t)}||_F^2 \leq \left(1 - \frac{3\epsilon^4 t}{6400}\right)||\mathbf{A}||_F^2. \tag{18}$$

We will prove this by induction on t. It is clearly true for $t = 0$ and for general t there are 2 possibilities:

(i)

$$|\mathbf{W}(S,T)| \leq \epsilon\sqrt{mn|S||T|} \tag{19}$$

for all $S \subseteq R$, $T \subseteq C$.
In this case the conditions of Theorem 1 are satisfied.

(ii) $\exists S \subseteq R$, $T \subseteq C$ with $|\mathbf{W}(S,T)| > \epsilon\sqrt{mn|S||T|}$.

We will show that in case (ii) we can find a $t+1$'st cut matrix so that (18) holds. Thus after at most $6400/(3\epsilon^4)$ iterations we will find ourselves in case (i).

Assume then that (19) is not satisfied and let $\mathbf{W} = \mathbf{W}^{(t)}$. We will describe a procedure which finds a pair R_{t+1}, C_{t+1} with

$$|\mathbf{W}(R_{t+1}, C_{t+1})| \geq \epsilon^2 mn/40. \tag{20}$$

The pair R_{t+1}, C_{t+1} will then be used to define $\mathbf{D}^{(t+1)}$.

We have for any subset S of the rows and any subset T of the columns,

$$\begin{aligned}
\mathbf{W}(S,T)^2 &= \left(\sum_{u\in S}\sum_{v\in T}\mathbf{W}(u,v)\right)^2 \\
&\leq |S|\sum_{u\in S}\left(\sum_{v\in T}\mathbf{W}(u,v)\right)^2 && \text{Cauchy-Schwartz} \\
&= |S|\sum_{u\in S}\sum_{v,v'\in T}\mathbf{W}(u,v)\mathbf{W}(u,v') \\
&= |S|\sum_{v\in T}\left(\sum_{u\in R}\mathbf{W}(u,v)\mathbf{W}(u,T)\right).
\end{aligned}$$

Defining

$$f(v) = \sum_{u\in S}\mathbf{W}(u,v)\mathbf{W}(u,T),$$

we see that

$$\sum_{v\in T} f(v) \geq \frac{\mathbf{W}(S,T)^2}{|S|}.$$

Also, it is easy to see that for each v, we have $f(v) \leq m|T|$. So if

$$Q = \left\{v \in C : f(v) \geq \frac{\mathbf{W}(S,T)^2}{2|S||T|}\right\}$$

then

$$|Q|m|T| + |T|\frac{\mathbf{W}(S,T)^2}{2|S||T|} \geq \frac{\mathbf{W}(S,T)^2}{|S|}$$

and so

$$|Q| \geq \frac{\mathbf{W}(S,T)^2}{2|S||T|m}. \tag{21}$$

Now choose a pair S,T which violate (19). ¿From (21) we see that for this S,T we have

$$|Q| \geq \epsilon^2 n/2.$$

Fix attention on this pair S,T and on a v in Q. Define a function $G = G_v : \mathbf{R} \rightarrow 2^R$ as follows:

$$G(\nu) = \begin{cases} \{u \in R : \mathbf{W}(u,v) \geq \nu\} & \text{if } \nu \geq 0 \\ \{u \in R : \mathbf{W}(u,v) \leq \nu\} & \text{if } \nu < 0 \end{cases} \tag{22}$$

It is easy to see from $a = \int_0^1 1_{x\leq a}dx$ etc. and $v \in Q$ that

$$\epsilon^2 mn/2 \leq f(v) = \int_0^1 \mathbf{W}(G(\nu),T)\ d\nu - \int_{-1}^0 \mathbf{W}(G(\nu),T)\ d\nu.$$

Thus for $v \in Q$, if ν is chosen uniformly at random from [-1,1]

$$\mathbf{E}_\nu(|\mathbf{W}(G(\nu),T)|) \geq \epsilon^2 mn/4.$$

Note next that $|\mathbf{W}(G(\nu),T)| \leq \sqrt{mn}||\mathbf{W}||_F \leq mn\ \forall\nu$ (using the inductive assumption (18) to bound $||\mathbf{W}||_F \leq ||\mathbf{A}||_F \leq \sqrt{mn}$). Let

$$\theta = \mathbf{Pr}(|\mathbf{W}(G(\nu),T)| \geq \epsilon^2 mn/8).$$

Then for $v \in Q$,

$$\epsilon^2 mn/4 \le \mathbf{E}_\nu(|\mathbf{W}(G(\nu),T)|) \le \theta mn + (1-\theta)\epsilon^2 mn/8$$

which implies $\theta \ge \epsilon^2/8$ or in other words that given $v \in Q$, if we pick ν at random in $[-1,1]$, with uniform density, then we have

$$\mathbf{Pr}\left(\{\mathbf{W}(G(\nu),T) \ge \epsilon^2 mn/8 \text{ and } \nu \ge 0\} \text{ OR } \{\mathbf{W}(G(\nu),T) \le -\epsilon^2 mn/8 \text{ and } \nu \le 0\}\right) \ge \epsilon^2/8. \tag{23}$$

Define

$$\begin{aligned} P_{\mathbf{W}}(R') &= \{x \in C: \ \mathbf{W}(R',x) \ge 0\} & N_{\mathbf{W}}(R') &= C \setminus P_{\mathbf{W}}(R') & \forall R' \subseteq R \\ P_{\mathbf{W}}(C') &= \{u \in R: \ \mathbf{W}(u,C') \ge 0\} & N_{\mathbf{W}}(C') &= R \setminus P_{\mathbf{W}}(C') & \forall C' \subseteq C. \end{aligned} \tag{24}$$

Equation (23) implies the following:

Lemma 1 *If $\exists S \subseteq R, T \subseteq C$ with $|\mathbf{W}(S,T)| \ge \epsilon\sqrt{mn|S||T|}$ and we pick v at random from C and ν at random (with uniform density from $[-1,1]$), then with probability at least $\epsilon^4/16$, we have*

$$\mathbf{W}(G(\nu), P_{\mathbf{W}}(G(\nu))) \ge \epsilon^2 mn/8 \quad \textit{OR} \quad \mathbf{W}(G(\nu), N_{\mathbf{W}}(G(\nu))) \le -\epsilon^2 mn/8. \tag{25}$$

□

We propose to use this lemma as follows: pick v, ν at random as above. **Check** whether (25) holds. (While T was unknown, both $P_{\mathbf{W}}(G(\nu))$ and $N_{\mathbf{W}}(G(\nu))$ are known once v, ν are.) If not, we repeat the trial a certain number of times. Whence we can argue that the probability of failure in all trials is low. Once we have v, ν satisfying (25) we can take $R_{t+1} = G(\nu)$ and $C_{t+1} = P_{\mathbf{W}}(G(\nu)))$ or $N_{\mathbf{W}}(G(\nu)))$. Then we can argue as in the proof of Theorem 7 that the Frobenius norm of our error matrix drops significantly. The catch is that checking whether $|\mathbf{W}(G(\nu), P_{\mathbf{W}}(G(\nu)))| \ge \epsilon^2 mn/8$, takes $O(n^2)$ time if naively done. We use sampling to do an approximate version of the check in time $\tilde{O}(\epsilon^{-12}\delta^{-1})$ below.

Steps 5-10 choose a random v, ν and try to see if (25) holds. $\tilde{R} = G(\nu)$ is represented by $\tilde{R} \cap U$ for a small random subset U. $\tilde{C}$ is defined as in (26) below. It is important to realise that $\tilde{R}, \tilde{C}$ are not explicitly computed. The value of $\mathbf{W}(\tilde{R}, \tilde{C})$ is estimated by $\tilde{W} = mn|\mathbf{W}(\tilde{R} \cap U_1, \tilde{C} \cap V_1))|/q^2$. Here U_1, V_1 are also small random subsets. This is done r_0 times and the best $\tilde{R}, \tilde{C}$ are re-checked in Step 11. If $|\tilde{W}|$ is large enough then we take steps (12,13) to ensure that $|\tilde{R}| \ge m/3, |\tilde{C}| \ge n/3$. This is used in the proof that the coefficient length of the decomposition is small.

We describe the algorithm and prove its correctness later. The constants in the algorithm are:

- $t_0 = \lceil 2500\epsilon^{-4} \rceil$.
- $p = \lceil 10^5 \epsilon^{-4} \log(6 r_0 s_0 t_0 \delta^{-1}) \rceil$.
- $r_0 = \lceil 32\epsilon^{-4} \rceil$.
- $s_0 = \lceil \log_2(3 t_0 \delta^{-1}) \rceil$.
- $q = 30 p r_0$.
- $q' = \lceil 30 p s_0 t_0 \delta^{-1} \rceil + \lceil 2 \times 10^8 \epsilon^{-8} \ln(12 s_0 t_0/\delta) \rceil$.

First algorithm to find a cut decomposition of A

1 For $t = 0, 1, \ldots t_0 - 1$ do:

2 Set $\mathbf{W}^{(t)} = \mathbf{A} - (\mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \ldots + \mathbf{D}^{(t)})$.

3 For $s = 1, 2, \ldots s_0$ do:

4 For $r = 1, 2, \ldots r_0$ do:

5 Pick a v from C uniformly at random.

6 Pick ν uniformly at random from $[-1, 1]$.

7 Pick random subsets U, U_1 of R independently with $|U| = p$ and $|U_1| = q$.

8 Pick a random subset V_1 of C independently with $|V_1| = q$.

9 $\tilde{R} \leftarrow G(\nu)$ and

$$\tilde{C} \leftarrow \begin{cases} P_{\mathbf{W}}(\tilde{R} \cap U) & \text{if } \nu > 0 \\ N_{\mathbf{W}}(\tilde{R} \cap U) & \text{if } \nu < 0 \end{cases} \tag{26}$$

10 Compute the following estimate $\tilde{W} = mn|\mathbf{W}(\tilde{R} \cap U_1, \tilde{C} \cap V_1)|/q^2$ of $\mathbf{W}(\tilde{R}, \tilde{C})$. Go to the next r.

11 Let $\tilde{R}, \tilde{C}$ refer to the largest value of $|\tilde{W}|$ found in the last execution of loop 4– 10.
Choose *new* random subsets $U_1 \subseteq R, V_1 \subseteq C$, $|U_1| = |V_1| = q'$ and recompute $\tilde{W}$ with q' replacing q.
If $|\tilde{W}| < \epsilon^2 mn/9$ goto the next s, unless $s = s_0$, in which case go to 15.

12 Compute the estimate ρ for $|\tilde{R}|$: $\rho = m|\tilde{R} \cap U_1|/q'$.
If $\rho \geq 2m/5$ goto 13, otherwise
Estimate $\mathbf{W}(R, \tilde{C})$ by $W_1 = mn\mathbf{W}(U_1, \tilde{C} \cap V_1)/{q'}^2$
If $W_1 \geq \epsilon^2 mn/19$ then $\tilde{R} \leftarrow R$, $\tilde{W} \leftarrow W_1$ and $\rho \leftarrow m$, otherwise
$\tilde{R} \leftarrow R \setminus \tilde{R}$, $\tilde{W} \leftarrow mnW(U_1, V_1)/{q'}^2 - W_1$ and $\rho \leftarrow m - \rho$.

13 Compute the estimate κ for $|\tilde{C}|$: $\kappa = n|\tilde{C} \cap V_1|/q'$.
If $\kappa \geq 2n/5$ goto 14, otherwise
Estimate $\mathbf{W}(\tilde{R}, C)$ by $W_2 = mn\mathbf{W}(\tilde{R} \cap U_1, V_1)/{q'}^2$
If $W_2 \geq \epsilon^2 mn/39$ then $\tilde{C} \leftarrow R$, $\tilde{W} \leftarrow W_2$ and $\kappa \leftarrow n$, otherwise
$\tilde{C} \leftarrow C \setminus \tilde{C}$, $\tilde{W} \leftarrow mnW(U_1, V_1)/{q'}^2 - W_2$ and $\kappa \leftarrow n - \kappa$.

14 $R_{t+1} \leftarrow \tilde{R}, C_{t+1} \leftarrow \tilde{C}, d_{t+1} \leftarrow \tilde{W}/\rho\kappa$ and

$$\mathbf{D}^{(t+1)} \leftarrow Cut(R_{t+1}, C_{t+1}.d_{t+1})$$

and go to the next t, unless $t = t_0$ in which case **FAIL**.

15 Terminate with $\mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \ldots \mathbf{D}^{(t)}$ as the approximation to $\mathbf{A}$.

The proof of correctness is based on the following sequence of lemmas. They show that the estimates are accurate enough with high enough probability.

Lemma 2 *Suppose* $\mathbf{W}$ *is an* $m \times n$ *matrix with set of rows* R *and set of columns* C. *Fix* $Y \subseteq R$. *Suppose* U *is a random subset of* R *of cardinality* p. *Then*

$$\mathbf{E}_U(\mathbf{W}(Y, P_{\mathbf{W}}(U \cap Y))) \geq \mathbf{W}(Y, P_{\mathbf{W}}(Y)) - \sqrt{\frac{mn}{p}}||\mathbf{W}||_F. \tag{27}$$

$$\mathbf{E}_U(\mathbf{W}(Y, N_{\mathbf{W}}(U \cap Y))) \leq \mathbf{W}(Y, N_{\mathbf{W}}(Y)) + \sqrt{\frac{mn}{p}}||\mathbf{W}||_F. \tag{28}$$

Proof We prove (27) only, as the proof of (28) is almost identical. Let $Z = P_{\mathbf{W}}(Y)$ and $Z' = P_{\mathbf{W}}(U \cap Y)$. We write

$$\mathbf{W}(Y, Z') = \mathbf{W}(Y, Z) - \mathbf{W}(Y, B_1) + \mathbf{W}(Y, B_2), \tag{29}$$

where

$$\begin{aligned} B_1 &= \{z \in C :\ \mathbf{W}(Y, z) > 0 \text{ and } \mathbf{W}(U \cap Y, z) < 0\}, \\ B_2 &= \{z \in C :\ \mathbf{W}(Y, z) < 0 \text{ and } \mathbf{W}(U \cap Y, z) > 0\}. \end{aligned}$$

Now if $X_z = \mathbf{W}(U \cap Y, z)$, $W_2(z) = \sum_{u \in Y} \mathbf{W}(u, z)^2$ then

$$X_z = \sum_{u \in Y} \mathbf{W}(u, z) \mathbf{1}_{u \in U}$$

and so

$$\mathbf{E}(X_z) = \frac{p}{m} \mathbf{W}(Y, z) \text{ and } \mathbf{Var}(X_z) \leq \frac{p}{m} W_2(z)$$

Hence, for any $\xi > 0$,

$$\mathbf{Pr}\left(\left| X_z - \frac{p}{m}\mathbf{W}(Y, z)\right| \geq \xi\right) \leq \frac{pW_2(z)}{m\xi^2} \tag{30}$$

If $z \in B_1$ then $X_z - (p/m)\mathbf{W}(Y, z) \leq -(p/m)\mathbf{W}(Y, z)$ and so applying (30) with $\xi = p\mathbf{W}(Y, z)/m$ we get that for each fixed z,

$$\mathbf{Pr}(z \in B_1) \leq \frac{mW_2(z)}{p\mathbf{W}(Y, z)^2}.$$

Thus,

$$\begin{aligned} \mathbf{E}\left(\sum_{z \in B_1} \mathbf{W}(Y, z)\right) &\leq \sum_{\{z \in C:\ \mathbf{W}(Y,z) > 0\}} \min\left\{\mathbf{W}(Y, z), \frac{mW_2(z)}{p\mathbf{W}(Y, z)}\right\} \\ &\leq \sum_{\{z \in C:\ \mathbf{W}(Y,z) > 0\}} \sqrt{\frac{mW_2(z)}{p}} \end{aligned} \tag{31}$$

By an identical argument we obtain

$$\mathbf{E}\left(\sum_{z \in B_2} \mathbf{W}(Y, z)\right) \geq -\sum_{\{z \in C:\ \mathbf{W}(Y,z) < 0\}} \sqrt{\frac{mW_2(z)}{p}}.$$

Hence, (using the Cauchy-Schwartz inequality),

$$\mathbf{E}(\mathbf{W}(Y, Z')) \geq \mathbf{W}(Y, Z) - \sum_{z \in C} \sqrt{\frac{mW_2(z)}{p}} \geq \mathbf{W}(Y, Z) - \sqrt{\frac{mn}{p}} ||\mathbf{W}||_F.$$

This proves (27) and (28) is proved similarly. $\square$

We use the lemma with $Y = \tilde{R} = G(\nu)$. Let $Z = P_{\mathbf{W}}(Y)$ if $\nu \geq 0$ and $Z = N_{\mathbf{W}}(Y)$ if $\nu < 0$, and let $Z' = \tilde{C}$ (as in Step 9 of the algorithm). From Lemma 2 and the Markov inequality applied to the *non-negative* random variable $\mathbf{W}(Y, P_{\mathbf{W}}(Y)) - \mathbf{W}(Y, P_{\mathbf{W}}(U \cap Y))$ we see that

$$\mathbf{Pr}\left(\mathbf{W}(Y, Z) - \mathbf{W}(Y, Z') \geq 2\sqrt{\frac{mn}{p}} ||\mathbf{W}||_F\right) \quad \leq \quad 1/2, \qquad \text{when } \nu > 0, \tag{32}$$

and similarly

$$\mathbf{Pr}\left(\mathbf{W}(Y,Z)-\mathbf{W}(Y,Z')\leq -2\sqrt{\frac{mn}{p}}||\mathbf{W}||_F\right) \quad\leq\quad 1/2, \qquad \text{when } \nu<0.$$

To aid the analysis, we define some events for each execution of the loop of Steps 4-10:

$$\begin{aligned}
E_1 &= \{\mathbf{W}(\tilde{R},P_{\mathbf{W}}(\tilde{R}))\geq \epsilon^2 mn/8\}\\
E_2 &= \{\mathbf{W}(\tilde{R},N_{\mathbf{W}}(\tilde{R}))\leq -\epsilon^2 mn/8\}\\
E_3 &= \{\mathbf{W}(\tilde{R},P_{\mathbf{W}}(\tilde{R}\cap U))\geq \mathbf{W}(\tilde{R},P_{\mathbf{W}}(\tilde{R}))-2\sqrt{mn/p}||\mathbf{W}||_F\}\\
E_4 &= \{\mathbf{W}(\tilde{R},N_{\mathbf{W}}(\tilde{R}\cap U))\leq \mathbf{W}(\tilde{R},N_{\mathbf{W}}(\tilde{R}))+2\sqrt{mn/p}||\mathbf{W}||_F\}
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{Pr}((E_1\wedge E_3)\vee(E_2\wedge E_4)) &\geq \mathbf{Pr}(E_1\wedge E_3)+\mathbf{Pr}(\neg E_1\wedge E_2\wedge E_4)\\
&= \mathbf{Pr}(E_3\mid E_1)\mathbf{Pr}(E_1)+\mathbf{Pr}(E_4\mid \neg E_1\wedge E_2)\mathbf{Pr}(\neg E_1\wedge E_2)\\
&\geq \mathbf{Pr}(E_1)/2+\left(\frac{\epsilon^4}{16}-\mathbf{Pr}(E_1)\right)\Big/2\\
&= \epsilon^4/32. \qquad (33)
\end{aligned}$$

for each execution of the Steps 4–10.

In the above calculation $\mathbf{Pr}(E_3\mid E_1)\geq 1/2$ follows from (32) – E_1 does not condition the random set U. Similarly, $\mathbf{Pr}(E_4\mid\neg E_1\wedge E_2)\geq 1/2$. Then $\mathbf{Pr}(\neg E_1\wedge E_2)=\mathbf{Pr}(E_1\vee E_2)-\mathbf{Pr}(E_1)$ and $\mathbf{Pr}(E_1\vee E_2)\geq \epsilon^4/16$ from Lemma 1.

The above shows that with sufficient probability, we "see" a pair $\tilde{R},\tilde{C}$ for which $|\mathbf{W}(\tilde{R},\tilde{C})|$ is large. We will now argue that with high probability, the estimated value $|mn\mathbf{W}(\tilde{R}\cap U_1,\tilde{C}\cap V_1)/q^2|$ and the real one – $|\mathbf{W}(\tilde{R},\tilde{C})|$ – are close so that we make no mistake. For this, we will need the definition of two other events.

$$\begin{aligned}
E_5 &= \left\{\left|\frac{mn}{q^2}\mathbf{W}(\tilde{R}\cap U_1,P_{\mathbf{W}}(\tilde{R}\cap U)\cap V_1)-\mathbf{W}(\tilde{R},P_{\mathbf{W}}(\tilde{R}\cap U))\right|\geq\sqrt{\frac{mn}{p}}||\mathbf{W}||_F\right\}\\
E_6 &= \left\{\left|\frac{mn}{q^2}\mathbf{W}(\tilde{R}\cap U_1,N_{\mathbf{W}}(\tilde{R}\cap U)\cap V_1)-\mathbf{W}(\tilde{R},N_{\mathbf{W}}(\tilde{R}\cap U))\right|\geq\sqrt{\frac{mn}{p}}||\mathbf{W}||_F\right\}.
\end{aligned}$$

The lemma below will bound the probability of E_5,E_6.

Lemma 3 *Suppose U_1,V_1 are random subsets of R,C respectively with $|U_1|=|V_1|=q$. Then, for any fixed $X\subseteq R$ and $Y\subseteq C$ we have*

$$\mathbf{Pr}\left(\left|\mathbf{W}(X,Y)-\frac{mn}{q^2}\mathbf{W}(X\cap U_1,Y\cap V_1)\right|\geq\sqrt{\frac{mn}{p}}||\mathbf{W}||_F\right)\leq\frac{3p}{q}. \qquad (34)$$

Proof Fix $X\subseteq R, Y\subseteq C$ and consider the random variable

$$Z=\mathbf{W}(X\cap U_1,Y\cap V_1)=\sum_{x\in X}\sum_{y\in Y}\xi_{x,y}$$

where

$$\xi_{x,y}=\mathbf{W}(x,y)1_{x\in U_1}1_{y\in V_1}.$$

Thus for all x,y, $\mathbf{E}(\xi_{x,y})=q^2\mathbf{W}(x,y)/mn$ and hence

$$\mathbf{E}(Z)=\frac{q^2\mathbf{W}(X,Y)}{mn}.$$

Now

$$\mathbf{E}(Z^2) = S_1 + S_2 + S_3 + S_4$$

where

$$\begin{aligned}
S_1 &= \frac{q^2}{mn}\sum_{x,y}\mathbf{W}(x,y)^2\\
S_2 &= \frac{q^2(q-1)}{mn(n-1)}\sum_{y\neq y',x}\mathbf{W}(x,y)\mathbf{W}(x,y')\\
&= \frac{q^2(q-1)}{mn(n-1)}\left(\sum_x \mathbf{W}(x,Y)^2 - \sum_{x,y}\mathbf{W}(x,y)^2\right)\\
S_3 &= \frac{q^2(q-1)}{m(m-1)n}\left(\sum_y \mathbf{W}(X,y)^2 - \sum_{x,y}\mathbf{W}(x,y)^2\right)\\
S_4 &= \frac{q^2(q-1)^2}{m(m-1)n(n-1)}\sum_{x\neq x',y\neq y'}\mathbf{W}(x,y)\mathbf{W}(x',y')\\
&= \frac{q^2(q-1)^2}{m(m-1)n(n-1)}\left(\mathbf{W}(X,Y)^2 - \sum_x \mathbf{W}(x,Y)^2 - \sum_y \mathbf{W}(X,y)^2 + \sum_{x,y}\mathbf{W}(x,y)^2\right).
\end{aligned}$$

Now

$$\sum_x \mathbf{W}(x,Y)^2 \leq n||\mathbf{W}||_F^2 \text{ and } \sum_y \mathbf{W}(X,y)^2 \leq m||\mathbf{W}||_F^2.$$

Hence,

$$\mathbf{Var}(Z) \leq \frac{3q^3}{mn}||\mathbf{W}||_F^2$$

and so for any $\xi > 0$ we have

$$\mathbf{Pr}(\left(\left|Z - \frac{q^2}{mn}\mathbf{W}(X,Y)\right| \geq \xi||\mathbf{W}||_F\right) \leq \frac{3q^3}{mn\xi^2}. \tag{35}$$

To obtain (34) we put $\xi = \frac{q^2}{(pmn)^{1/2}}$ in (35). □

Consider an execution of Steps 4–10 i.e. a fixed t, s. This will be considered *successful* if $(E_1 \wedge E_3) \vee (E_2 \wedge E_4)$ occurs at least once and $E_5 \vee E_6$ never occurs. In this case the values of $\tilde{R}, \tilde{C}$ passed onto Step 11 will satisfy

$$|\mathbf{W}(\tilde{R},\tilde{C})| \geq \epsilon^2 mn/8 \text{ and } |\tilde{W} - \mathbf{W}(\tilde{R},\tilde{C})| \leq 3mn/\sqrt{p}. \tag{36}$$

We see from (33) and Lemma 3 that

$$\begin{aligned}
\mathbf{Pr}(\text{Steps 4–10 are successful}) &\geq \left(1-\left(1-\frac{\epsilon^4}{32}\right)^{r_0}\right)\left(1-\frac{3pr_0}{q}\right)\\
&\geq 1/2.
\end{aligned}$$

So the probability that none of the $\tilde{R}, \tilde{C}$ etc. passed to Step 11 satisfy (36) is at most $2^{-s_0} \leq \delta/(3t_0)$.

We next observe that it follows from Lemma 3 that with probability at least $1 - \delta/(3t_0)$ *all* of the estimates $\tilde{W}$ made in Step 11 and W_1, W_2 made in Steps 12,13 are accurate to within $\sqrt{mn/p}||\mathbf{W}||_F \leq \epsilon^2 mn/10000$.

We now consider the accuracy of the estimates ρ, κ in Steps 12,13.

Lemma 4

$$\begin{aligned}\mathbf{Pr}(|\rho - |\tilde{R}|| \geq \epsilon^4 mn/3000) &\leq \frac{\delta}{6s_0t_0} \\ \mathbf{Pr}(|\kappa - |\tilde{C}|| \geq \epsilon^4 mn/3000) &\leq \frac{\delta}{6s_0t_0}\end{aligned}$$

Proof We need only deal with ρ, κ as produced in the first statements of Steps 12,13. Applying the results of Section 6 of Hoeffding [20] (sampling with replacement) we see that for any $\xi > 0$

$$\mathbf{Pr}(||\tilde{R}| - m\rho/q'| \geq \xi m/q') \leq 2\exp\{-2\xi^2/q'\}.$$

Putting $\xi = \epsilon^4 q'/3000$ we see that

$$\mathbf{Pr}(||\tilde{R}| - m\rho/q'| \geq \epsilon^4 m/3000) \leq \frac{\delta}{6s_0t_0}.$$

Similarly,

$$\mathbf{Pr}(||\tilde{C}| - n\kappa/q'| \geq \epsilon^4 n/3000) \leq \frac{\delta}{6s_0t_0}.$$

□

We summarise what we want from Lemma 2, 3 and 4.

Lemma 5 *For each fixed t, with probability at least $1 - 2\delta/3t_0$:*

- *If $\exists S, T$ with $|\mathbf{W}(S,T)| \geq \epsilon mn$ then the algorithm returns R_{t+1}, C_{t+1} with*

$$|\mathbf{W}(R_{t+1}, C_{t+1})| \geq \frac{\epsilon^2 mn}{40}. \quad (37)$$

- *If the algorithm returns a pair R_{t+1}, C_{t+1} then (37) holds.*
- $|\tilde{W} - \mathbf{W}(R_{t+1}, C_{t+1})| \leq \frac{\epsilon^2 mn}{1000}$.
- $|R_{t+1}| \geq \frac{m}{3}, |\rho - |R_{t+1}|| \leq \frac{\epsilon^4 m}{3000}$.
- $|C_{t+1}| \geq \frac{n}{3}, |\kappa - |C_{t+1}|| \leq \frac{\epsilon^4 n}{3000}$.

□

¿From Lemmas 5 and 4 we observe that with probability at least $1 - \delta$ the following holds throughout the algorithm:

$$\left|\frac{\mathbf{W}(R_{t+1}, C_{t+1})}{\tilde{W}} - 1\right| \leq \frac{1}{10}.$$

$$\left|\frac{|R_{t+1}||C_{t+1}|}{\rho\kappa} - 1\right| \leq \frac{1}{10}.$$

In which case, if

$$\tilde{d}_{t+1} = \frac{\mathbf{W}(R_{t+1}, C_{t+1})}{|R_{t+1}|\,|C_{t+1}|} \quad (38)$$

then $\delta_{t+1} = d_{t+1} - \tilde{d}_{t+1}$ satisfies

$$|\delta_{t+1}| \leq |\tilde{d}_{t+1}|/2. \quad (39)$$

Finally note that

$$\begin{aligned}
\sum_{i,j}((\mathbf{W}-\mathbf{D}^{(t+1)})(i,j))^2 - \sum_{i,j}\mathbf{W}(i,j)^2 &= \sum_{i\in R_{t+1},j\in C_{t+1}}((\mathbf{W}(i,j)-\tilde{d}_{t+1}-\delta_{t+1})^2-\mathbf{W}(i,j)^2) \\
&= -|R_{t+1}||C_{t+1}|\tilde{d}_{t+1}^2 + |R_{t+1}||C_{t+1}|\delta_{t+1}^2 \qquad (40) \\
&\le -3|R_{t+1}||C_{t+1}|\tilde{d}_{t+1}^2/4 \\
&= -3\mathbf{W}(R_{t+1},C_{t+1})^2/(4|R_{t+1}||C_{t+1}|) \\
&\le -3\epsilon^4 mn/6400, \qquad (41)
\end{aligned}$$

which establishes (18).

We now deal with the coefficient length of the decomposition. Arguing from (39) and (40) we see that

$$\begin{aligned}
||\mathbf{W}-\mathbf{D}^{(t+1)}||_F^2 - ||\mathbf{W}||_F^2 &= -|R_{t+1}|\,|C_{t+1}|d_{t+1}(\bar{d}_{t+1}-\delta_{t+1}) \\
&\le -|R_{t+1}|\,|C_{t+1}|d_{t+1}^2/3.
\end{aligned}$$

Consequently,

$$\frac{1}{3}\sum_{t=1}^{s}|R_t|\,|C_t|d_t^2 \le ||\mathbf{A}||_F^2. \qquad (42)$$

Our bound on the coefficient length follows from $|R_t| \ge m/3$ and $|C_t| \ge n/3$. □

4.3 Second Algorithm

Proof of Theorem 2. At a general stage, for some $t \ge 0$, we will, with sufficently high probability, have found cut matrices $\mathbf{D}^{(1)}, \dots \mathbf{D}^{(t)}$ such that $\mathbf{W}^{(t)} = \mathbf{A} - (\mathbf{D}^{(1)} + \cdots + \mathbf{D}^{(t)})$ satisfies

$$||\mathbf{W}^{(t)}||_F^2 \le \left(1 - \frac{3\epsilon^2 t}{192}\right)||\mathbf{A}||_F^2. \qquad (43)$$

We will prove this by induction on t. It is clearly true for $t = 0$.

In the following we let $\mathbf{W} = \mathbf{W}^{(t)}$. We will describe a procedure which either determines that $||\mathbf{W}||_C \le \epsilon\sqrt{mn}||\mathbf{A}||_F$ or finds a subset Y_{t+1} of R and a subset Z_{t+1} of C such that

$$|\mathbf{W}(P_{\mathbf{W}}(Z_{t+1}), P_{\mathbf{W}}(Y_{t+1}))| \ge \epsilon\sqrt{mn}||\mathbf{A}||_F/4. \qquad (44)$$

The idea of the algorithm is as follows: Suppose $\mathbf{W}(S,T)$ is large and positive for some S,T. We choose random p-sets $U \subseteq R, V \subseteq C$. By enumerating all subsets of V we will eventually, (without knowing it) come across $V' = T \cap V$. Let $S' = P_{\mathbf{W}}(V')$. By enumerating all subsets of U we will come across $U' = U \cap S'$. Let $T' = P_{\mathbf{W}}(U')$. We show that $\mathbf{W}(S',T')$ is likely to be large. This is Steps 5-7. Step 8 checks the best looking pair U', V' found in Steps 5-7. Steps 9 and 10 boost the sizes of our guesses S', T' for R_{t+1}, C_{t+1}. This is needed to prove a bound on the coefficient length of the decomposition.

The constants in the algorithm are:

- $t_0 = \lceil \frac{64}{3\epsilon^2} \rceil$.
- $r_0 = \lceil \log_2(3t_0/\delta) \rceil$

- $p = \lceil 10^5 \epsilon^{-2} + 2\log_2 1/\delta \rceil$.
- $q = 30p2^p$.

Second algorithm to find an approximation to A

1 For $t = 0, 1, \ldots, t_0 - 1$ do:

2 Set $\mathbf{W}^{(t)} = \mathbf{A} - (\mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \ldots \mathbf{D}^{(t)})$.

3 For $r = 1, 2, \ldots, r_0$ do

4 Independently choose random subsets $U, U_1 \subseteq R$, $V, V_1 \subseteq C$ with $|U| = |V| = p$ and $|U_1| = |V_1| = q$.

5 For all $U' \subseteq U$ and $V' \subseteq V$

6 Compute an estimate $\tilde{W} = mn\mathbf{W}(P_{\mathbf{W}}(V') \cap U_1, P_{\mathbf{W}}(U') \cap V_1)/q^2$ of $\mathbf{W}(P_{\mathbf{W}}(V'), P_{\mathbf{W}}(U'))$

7 Search for R_{t+1}, C_{t+1} giving a large negative value of $\mathbf{W}$. Analogous to 5-6 but with $P_{\mathbf{W}}$ replaced by $N_{\mathbf{W}}$ etc.

8 Let $U', V', \tilde{R} \leftarrow P_{\mathbf{W}}(V'), \tilde{C} = P_{\mathbf{W}}(U')$ (or $\tilde{R} \leftarrow N_{\mathbf{W}}(V'), \tilde{C} = N_{\mathbf{W}}(U')$) refer to the largest value of $|\tilde{W}|$ found in the previous execution of loop 5-7.
Choose new random values for U_1, V_1 and recompute $\tilde{W}$.
If $|\tilde{W}| < 3\epsilon\sqrt{mn}||\mathbf{A}||_F/4$ then go to the next r, (unless $r = r_0$, in which case go to Step 12) otherwise
Compute the following estimates ρ, κ for $|P_{\mathbf{W}}(V')|$ $|P_{\mathbf{W}}(U')|$ respectively:
$\rho \leftarrow m|\{u \in U : \mathbf{W}(u, V') \geq 0\}|/p$ and $\kappa \leftarrow n|\{v \in V : \mathbf{W}(U', v) \geq 0\}|/p$.
[Remark: we now boost the sizes of $\tilde{R}, \tilde{C}$ – needed to prove our bound on the coefficient length of the decomposition).]
If $\tilde{W} < 0$ then go to 11, otherwise

9 If $\rho \geq 2m/5$ go to 10, otherwise
Estimate $\mathbf{W}(R, \tilde{C})$ by $W_1 = mn\mathbf{W}(U_1, \tilde{C} \cap V_1)/q^2$.
If $W_1 \geq 3\epsilon\sqrt{mn}||\mathbf{A}||_F/8$ then let $\tilde{R} = R$, $\tilde{W} = W_1$ and $\rho \leftarrow m$,
otherwise let $\tilde{R} = R \setminus P_{\mathbf{W}}(V')$, $\tilde{W} = mn\mathbf{W}(U_1, V_1) - W_1$ and $\rho = m - \rho$.

10 If $\kappa \geq 2n/5$ go to the next t, otherwise
Estimate $\mathbf{W}(\tilde{R}, C)$ by $W_2 = mn\mathbf{W}(\tilde{R} \cap U_1, V_1)/q^2$.
If $W_2 \geq 3\epsilon\sqrt{mn}||\mathbf{A}||_F/16$ then $\tilde{C} \leftarrow C$, $\tilde{W} \leftarrow W_2$ and $\kappa \leftarrow n$,
otherwise $\tilde{C} \leftarrow C \setminus P_{\mathbf{W}}(U')$, $\tilde{W} \leftarrow mn\mathbf{W}(U_1, V_1) - W_2$ and $\kappa \leftarrow n - \kappa$.
Set $R_{t+1} \leftarrow \tilde{R}, C_{t+1} \leftarrow \tilde{C}, d_{t+1} = \tilde{W}/(\rho\kappa)$ and

$$\mathbf{D}^{(t+1)} \leftarrow Cut(R_{t+1}, C_{t+1}.d_{t+1})$$

and go to the next t, unless $t = t_0$ in which case **FAIL**.

11 Similar to 9,10.

12 Terminate with $\mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \ldots \mathbf{D}^{(t)}$ as the approximation to $\mathbf{A}$.

The proof of correctness is based on the following sequence of lemmas:

Lemma 6 *Suppose there exist $S \subseteq R$, $T \subseteq C$ such that $|\mathbf{W}(S,T)| \geq \epsilon\sqrt{mn}||\mathbf{A}||_F$. Then with probability at least $3/4$ we have*

$$|\mathbf{W}(S', T')| \geq |\mathbf{W}(S,T)| - \epsilon\sqrt{mn}||\mathbf{A}||_F/100 \tag{45}$$

where $S' = P_{\mathbf{W}}(T \cap V)$ and $T' = P_{\mathbf{W}}(S' \cap U)$ or $S' = N_{\mathbf{W}}(T \cap V)$ and $T' = N_{\mathbf{W}}(S' \cap U)$.

Proof Let S,T maximise $\mathbf{W}(S,T)$. Then

$$\mathbf{E}_{U,V}(\mathbf{W}(S,T)-\mathbf{W}(S',T'))=\mathbf{E}_V(\mathbf{W}(S,T)-\mathbf{W}(S',T))+\mathbf{E}_V(\mathbf{E}_U(\mathbf{W}(S',T)-\mathbf{W}(S',T')\mid V)).$$

It follows from Lemma 2 and (43) that

$$\begin{aligned}\mathbf{E}_V(\mathbf{W}(S,T)-\mathbf{W}(S',T)) &\le \sqrt{mn/p}||\mathbf{A}||_F,\\ \mathbf{E}_U(\mathbf{W}(S',T)-\mathbf{W}(S',T')\mid V) &\le \sqrt{mn/p}||\mathbf{A}||_F,\end{aligned}$$

and so

$$\mathbf{E}_{U,V}(\mathbf{W}(S,T)-\mathbf{W}(S',T'))\le 2\sqrt{mn/p}||\mathbf{A}||_F.$$

Now $\mathbf{W}(S,T)-\mathbf{W}(S',T')\ge 0$ and so by the Markov inequality

$$\mathbf{Pr}(\mathbf{W}(S,T)-\mathbf{W}(S',T')\ge 16\sqrt{mn/p}||\mathbf{A}||_F)\le 1/8.$$

A similar argument deals with large negative values of $\mathbf{W}(S,T)$. □

We observe next that at some time during the enumeration of the subsets of U,V we will have $U'=U\cap S'$ and $V'=V\cap T$. We say that the loop Steps 5-7 is *successful* if (45) holds for these values $U'=U\cap S', V'=V\cap T$ and

$$\left|\mathbf{W}(X,Y)-\frac{mn}{q^2}\mathbf{W}(X\cap U_1,Y\cap V_1)\right|\le\sqrt{\frac{mn}{p}}||\mathbf{A}||_F$$

for all $X=P_{\mathbf{W}}(U'),Y=P_{\mathbf{W}}(V')$ and for all $X=N_{\mathbf{W}}(U'),Y=N_{\mathbf{W}}(V')$. Applying Lemmas 3 and 6 we see that

$$\begin{aligned}\mathbf{Pr}(\text{Steps 5–7 are successful}) &\ge \frac{3}{4}\left(1-\frac{3p2^p}{q}\right)\\ &\ge 1/2.\end{aligned}$$

So the probability that none of the $\tilde{R},\tilde{C}$ etc. passed to Step 8 satisfy (45) is at most $2^{-r_0}\le\delta/(3t_0)$.

We next observe that it follows from Lemma 3 that with probability at least $1-\delta/3$ *all* of the new estimates $\tilde{W}$ made in Step 8 and all of the estimates W_1,W_2 made in Steps 9,10,11 are accurate to within $\sqrt{mn/p}||\mathbf{A}||_F\le\epsilon\sqrt{mn}||\mathbf{A}||_F/100$.

So with probability at least $1-\delta/t_0$ the outputs $R_{t+1},C_{t+1},\rho,\kappa$ satisfy

$$\begin{aligned}|\mathbf{W}(R_{t+1},C_{t+1})| &\ge \epsilon\sqrt{mn}||\mathbf{A}||_F/8\\ \frac{|R_{t+1}|}{m},\frac{|C_{t+1}|}{n} &\ge 1/3\\ \left|\frac{\rho}{|R_{t+1}|}-1\right|,\left|\frac{\kappa}{|C_{t+1}|}-1\right| &\le .03\end{aligned}\tag{46}$$

We deduce that (39) holds for $\bar{d}_{t+1}$ as defined in (38).

¿From which it follows that

$$\sum_{i,j}((\mathbf{W}-\mathbf{D}^{(t+1)})(i,j))^2-\sum_{i,j}\mathbf{W}(i,j)^2\le -3\epsilon^2||\mathbf{A}||_F^2/192.$$

This verifies the inductive hypothesis (43).

The bound on the coefficient length of the decomposition is proved as in the first algorithm. In particular, (42) still holds. □

4.4 Maximising $|\mathbf{W}(S,T)|$ approximately

We can easily modify the second algorithm to find a pair S, T that approximately maximises $|\mathbf{W}(S,T)|$.

Theorem 8 *Let $\mathbf{A}, \epsilon, \delta$ be as in Theorem 2. Then with probability at least $1-\delta$ we can in time $2^{\tilde{O}(1/\epsilon^2)}/\delta^2$, find $S \subseteq R, T \subseteq C$ such that*

$$|\mathbf{A}(S,T)| \geq |\mathbf{A}(X,Y)| - \epsilon\sqrt{mn}||\mathbf{A}||_F \qquad \forall X \subseteq R, Y \subseteq C.$$

Proof We simply execute Steps 3-11 once (i.e. take $t_0 = 1$). If $\max|\mathbf{A}(X,Y)| = \alpha mn, \alpha \geq \epsilon$ then with probability at least $1-\delta$ we can find S, T with $|\mathbf{A}(S,T)| \geq (\alpha - \epsilon)\sqrt{mn}||\mathbf{A}||_F$ – in our proof we show that we make an *additive* error of at most $\epsilon\sqrt{mn}||\mathbf{A}||_F$.

5 Partitions

As previously mentioned, an earlier paper [15], was concerned with the algorithmic uses of a certain partition of the vertex sets of graphs and hypergraphs. We now show how such a partition can be recovered quickly from our matrix decomposition.

Let $G = (V, E)$ be a graph with n vertices and let $\mathbf{A}$ be its adjacency matrix. For disjoint sets $A, B \subseteq V$ let $e(A, B)$ denote the number of edges between A and B. The *density* $d(A, B)$ is defined by

$$d(A,B) = \frac{e(A,B)}{|A|\,|B|}.$$

We let $d(A,A) = e(A,A)/\binom{|A|}{2}$. A disjoint pair $A, B \subseteq V$ is said to be $\epsilon - regular$ if for *every* $X \subseteq A$ with $|X| \geq \epsilon|A|$ and $Y \subseteq B$ with $|Y| \geq \epsilon|B|$, we have

$$|d(X,Y) - d(A,B)\,| < \epsilon.$$

5.1 Pseudo-Regular Partitions

Let $\mathcal{P} = V_1, \ldots, V_k$ be a partition of V. Let $d_{i,j} = d(V_i, V_j)$. For $X \subseteq V$ and $I \subseteq K = \{1, 2, \ldots, k\}$ we let $X_I = \bigcup_{i \in I} X_i$ where $X_i = X \cap V_i$. For *disjoint* subsets S, T of V let

$$\Delta_{\mathcal{P}}(S,T) = e(S,T) - \sum_{i \in K}\sum_{j \in K} d_{i,j}|S_i||T_j|.$$

The term $d_{i,j}|S_i||T_j|$ would be (approximately) $e(S_i, T_j)$ if the pair V_i, V_j were ϵ-regular. So $\Delta_{\mathcal{P}}(S,T)$ measures the total deviation from regularity.

A partition $\mathcal{P}$ is *ϵ-pseudo-regular* if

$$|\Delta_{\mathcal{P}}(S,T)| \leq \epsilon n^2 \text{ for all disjoint subsets } S, T \text{ of } V.$$

Notice that we do not insist on the subsets being of (almost) the same size. This can easily be enforced, at a small extra cost, see Section 5.1.1 below.

The reader will observe that if $\mathcal{P}$ is ϵ-pseudo-regular then for every disjoint pair $S, T \subseteq V$, $e(S,T)$ is almost determined by the values $|S_i|, |T_j|$. Thus we can for example approximately solve Max-Cut by choosing values for $|S_i|, |T_j|$ which approximately maximise $\sum_{i \in K}\sum_{j \in K} d_{i,j}|S_i||T_j| +$

$\sum_{i \in K} d_{i,i}|S_i|T_i|$. This was the approach taken in [15] viz. compute an ϵ-pseudo-regular partition and then proceed as indicated.

We show next how we can obtain such a partition from our matrix decomposition algorithms.

For a partition $\mathcal{Q} = W_1, W_2, \ldots, W_q$ and $V \times V$ matrix $\mathbf{M}$ we define $\mathbf{M}_\mathcal{Q}$ by $\mathbf{M}_\mathcal{Q}(p,q) = \mathbf{M}(W_i, W_j)/(|W_i||W_j|)$ for $(p,q) \in W_i \times W_j$. Thus for *disjoint* S, T

$$\begin{aligned} \mathbf{A}(S,T) &= e(S,T), \\ \mathbf{A}_\mathcal{Q}(S,T) &= \sum_{i \in K} \sum_{j \in K} d_{i,j}|S_i||T_j|, \end{aligned}$$

and so

$$\mathbf{A}(S,T) - \mathbf{A}_\mathcal{Q}(S,T) = \Delta_\mathcal{Q}(S,T). \tag{47}$$

$S \subseteq V$ is said to be *compatible* with $\mathcal{Q}$ if $S = \bigcup_{i \in I} W_i$ for some $I \subseteq K$. A matrix $\mathbf{M}$ is said to be *compatible* with $\mathcal{Q}$ if $\mathbf{M}(p,q)$ is constant over $W_i \times W_j$ for all $i, j \in K$.

Lemma 7 **(a)** *Let partition $\mathcal{Q} = W_1, W_2, \ldots, W_\ell$ be a refinement of partition $\mathcal{P}$. If $\mathbf{M}$ is compatible with $\mathcal{P}$ then*

$$\sup_{S,T \subseteq V} |\mathbf{A}(S,T) - \mathbf{A}_\mathcal{Q}(S,T)| \leq 2 \sup_{S,T \subseteq V} |\mathbf{A}(S,T) - \mathbf{M}(S,T)|.$$

(b)

$$||\mathbf{A} - \mathbf{A}_\mathcal{P}||_F \leq ||\mathbf{A} - \mathbf{M}||_F.$$

Proof (a) Let $m_{i,j}$ denote the common value of $\mathbf{M}(p,q)$ for $(p,q) \in W_i \times W_j$. Then for disjoint $S, T \subseteq V$,

$$|\mathbf{A}_\mathcal{Q}(S,T) - \mathbf{M}(S,T)| = \left| \sum_{i,j} (d(W_i, W_j) - m_{i,j})|S \cap W_i|\,|T \cap W_j| \right|. \tag{48}$$

Keeping S fixed we see that the extremal values of the RHS of (48) are obtained for T which are compatible with $\mathcal{Q}$. Indeed to maximise the sum we would put $T \cap W_j = W_j$ if $\sum_i (d(W_i, W_j) - m_{i,j})|S \cap W_i| > 0$ and $T \cap W_j = \emptyset$ otherwise. Similarly, for a fixed T we should choose S which is compatible with $\mathcal{Q}$. But when S, T are both compatible with $\mathcal{Q}$ we find that $\mathbf{A}_\mathcal{Q}(S,T) = \mathbf{A}(S,T)$ and so

$$\sup_{\substack{S,T \subseteq V \\ S \cap T = \emptyset}} |\mathbf{A}_\mathcal{Q}(S,T) - \mathbf{M}(S,T)| \leq \sup_{\substack{S,T \subseteq V \\ S \cap T = \emptyset}} |\mathbf{A}(S,T) - \mathbf{M}(S,T)|$$

and (a) follows from

$$\sup_{\substack{S,T \subseteq V \\ S \cap T = \emptyset}} |\mathbf{A}(S,T) - \mathbf{A}_\mathcal{Q}(S,T)| \leq \sup_{\substack{S,T \subseteq V \\ S \cap T = \emptyset}} |\mathbf{A}(S,T) - \mathbf{M}(S,T)| + \sup_{S,T \subseteq V} |\mathbf{A}_\mathcal{Q}(S,T) - \mathbf{M}(S,T)|.$$

(b) Now let $m_{i,j}$ denote the common value of $\mathbf{M}(p,q)$ for $(p,q) \in V_i \times V_j$. Then

$$\begin{aligned} ||\mathbf{A} - \mathbf{M}||_F^2 - ||\mathbf{A} - \mathbf{A}_\mathcal{P}||_F^2 &= \sum_{i,j} \sum_{(p,q) \in V_i \times V_j} ((\mathbf{A}(p,q) - m_{i,j})^2 - (\mathbf{A}(p,q) - d(V_i, V_j))^2) \\ &= \sum_{i,j} |V_i|\,|V_j|(m_{i,j} - d(V_i, V_j))^2 \qquad (49) \\ &\geq 0. \end{aligned}$$

$\square$

Returning to the problem of computing an ϵ-pseudo-regular partition, let $\mathbf{D}^{(1)}, \mathbf{D}^{(2)}, \ldots, \mathbf{D}^{(s)}$ be cut matrices as defined in Theorem 2. Let $V_1, V_2, \ldots, V_k$, $k \leq 4^s$ be the coarsest partition of V into subsets such that each R_t or C_t is the union of subsets of the partition.

Let $\mathbf{D} = \mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \cdots + \mathbf{D}^{(s)}$ and note that $\mathbf{D}$ is compatible with $\mathcal{P}$.

We claim that:

$$\text{Partition } V_1, V_2, \ldots V_k \text{ is } 2\epsilon\text{-pseudo-regular.} \tag{50}$$

Applying Lemma 7(a) with $\mathcal{Q} = \mathcal{P}$ and $\mathbf{M} = \mathbf{D}$ we see that

$$\begin{aligned}
\sup_{\substack{S,T\subseteq V\\ S\cap T=\emptyset}} |\Delta_{\mathcal{P}}(S,T)| &= \sup_{\substack{S,T\subseteq V\\ S\cap T=\emptyset}} |\mathbf{A}(S,T) - \mathbf{A}_{\mathcal{P}}(S,T)| \qquad \text{by (47)}\\
&\leq 2 \sup_{\substack{S,T\subseteq V\\ S\cap T=\emptyset}} |\mathbf{A}(S,T) - \mathbf{D}(S,T)|\\
&= 2 \sup_{\substack{S,T\subseteq V\\ S\cap T=\emptyset}} |\bar{\mathbf{W}}(S,T)|\\
&\leq 2\epsilon n^2,
\end{aligned}$$

and (50) follows.

5.1.1 Equitable Partitions

Let a partition $\mathcal{P} = V_1, V_2, \ldots, V_k$ of V be *equitable* if $||V_i| - |V_j|| \leq 1$ for all i, j. The decomposition in Szemerédi's theorem can be assumed to be equitable (see Theorem 9 below). We show that equitability can be achieved at a small extra cost.

After finding an ϵ-pseudo-regular partition $\mathcal{P}$ as described above we take each V_i and partition it into $V_{i,j}$, $1 \leq j \leq s_i$ where $|V_{i,j}| = \lfloor \epsilon n/(10k) \rfloor$ for $1 \leq j < s_i$ and $|V_{i,s_i}| < \epsilon n/(10k)$ to obtain a partition $\mathcal{Q}$ which is a refinement of $\mathcal{P}$. Applying (50) and Lemma 7(a) with $\mathbf{M} = \mathbf{A}_{\mathcal{P}}$ we see that

$$\sup_{\substack{S,T\subseteq V\\ S\cap T=\emptyset}} |\Delta_{\mathcal{Q}}(S,T)| \leq 2 \sup_{S,T\subseteq V} |\Delta_{\mathcal{P}}(S,T)| \leq 2\epsilon n^2.$$

Now if $R = \bigcup_{i\in K} V_{i,s_i}$ then $|R| \leq \epsilon n/10$ and so if we equitably spread the vertices in R over the other subsets of $\mathcal{Q}$ we will obtain an equitable partition $\mathcal{Q}'$, say, with $\sup_{\substack{S,T\subseteq V\\ S\cap T=\emptyset}} |\Delta_{\mathcal{Q}'}(S,T)| \leq 3\epsilon n^2$.

In some circumstances we want k to be at least a certain amount k_0. In this case we simply replace $\lfloor \epsilon n/k \rfloor$ by $\min\{\lfloor \epsilon n/k \rfloor, \lfloor n/k_0 \rfloor\}$.

Finally note that if we want to partition a digraph (or bipartite graph) in an analogous way, then we can dispense with the symmetric construction (3) and use Theorem 2 directly.

5.2 Szemerédi's partition

Theorem 9 *(***Szemerédi's Regularity Lemma***) For every $\epsilon > 0$ and integer $m > 0$ there are integers $P(\epsilon, m), Q(\epsilon, m)$ with the following property: for* every *graph $G = (V, E)$ with $n \geq P(\epsilon, m)$ vertices there is a partition of V into $k+1$ classes $V_0, V_1, \ldots, V_k$ such that*

- *$m \leq k \leq Q(\epsilon, m)$.*

- $|V_0| \leq \epsilon n$.
- $|V_1| = |V_2| = \cdots = |V_k|$.
- *All but at most* ϵk^2 *of the pairs* (V_i, V_j) *are* ϵ*-regular.*

The partition alluded to in the theorem will be referred to as an $\epsilon - RL$ partition. V_0 is the *exceptional* set. We avoided an exceptional set in Section 5.1.1 but find it convenient to use one here. When discussing ϵ-RL partitions, we will use equitable to denote the second two conditions.

As mentioned previously, Szemerédi's proof is non-constructive but Alon et al show how to construct an ϵ-RL partition (with different values of P, Q) in time $O(\alpha_1(\epsilon)M(n))$. where $M(n)$ is the time needed to multiply two $n \times n$ 0-1 matrices.

We now give an alternative proof to [1] of

Theorem 10 *An* ϵ*-RL partition is computable in polynomial time. (We can in fact produce an implicit description in time dependend only on* ϵ*).*

Proof We start with an arbitrary equitable partition $\mathcal{P} = (V_0 = \emptyset, V_1, \ldots, V_k)$ with $k = \lceil 1/\epsilon \rceil$. Given a partition $\mathcal{P}$ we define the $(V \setminus V_0) \times (V \setminus V_0)$ matrix $\mathbf{A} \circ \mathcal{P}$ to be the submatrix $\mathbf{A} - \mathbf{A}_{\mathcal{P}}$ obtained by deleting rows and columns corresponding to V_0.

We will show that if $\mathcal{P}$ is not ϵ-RL then we can find a new equitable partition $\mathcal{P}'$ with $k\epsilon^{-1} \leq k' \leq k^3 4^{2k} \epsilon^{-1}$ subsets such that

$$||\mathbf{A} \circ \mathcal{P}'||_F^2 \leq ||\mathbf{A} \circ \mathcal{P}||_F^2 - \epsilon^7 n^2/100. \tag{51}$$

The process stops after at most $100/\epsilon^7$ iterations.

Let $I = \{(i,j) :\ V_i, V_j \text{ is not } \epsilon\text{-regular}\}$. Suppose $|I| \geq \epsilon k^2$. For each $(i,j) \in I$ we consider the corresponding $V_i \times V_j$ submatrix $\mathbf{A}_{i,j}$ of $\mathbf{A} \circ \mathcal{P}$. We note that $(i,j) \in I$ implies $||\mathbf{A}_{i,j}||_C \geq \epsilon^3 \lfloor (1-\epsilon)n/k \rfloor^2$. We use the second decomposition algorithm to construct a matrix $\mathbf{D}_{i,j} = CUT(R_{i,j}, C_{i,j}, \delta_{i,j})$ such that say,

$$||\mathbf{A}_{i,j} - \mathbf{D}_{i,j}||_F^2 \leq ||\mathbf{A}_{i,j}||_F^2 - \epsilon^6 (n/k)^2/100. \tag{52}$$

Let $\mathbf{D} = \sum_{(i,j) \in I} \mathbf{D}_{i,j}$ and let $\mathcal{Q} = W_1, W_2, \ldots, W_\ell$ be the coarsest refinement of $\mathcal{P}$ such that each $R_{i,j}$ and $C_{i,j}$ are the union of members of $\mathcal{Q}$. We observe that

(i) $||\mathbf{A} \circ \mathcal{P} - \mathbf{D}||_F^2 \leq ||\mathbf{A} \circ \mathcal{P}||_F^2 - \epsilon^7 n^2/100$.

(ii) $||\mathbf{A} \circ \mathcal{Q}||_F = ||(\mathbf{A} \circ \mathcal{P}) - (\mathbf{A} \circ \mathcal{P})_{\mathcal{Q}}||_F \leq ||\mathbf{A} \circ \mathcal{P} - \mathbf{D}||_F$.

(iii) $\ell \leq k4^k$.

Inequality (i) follows from the fact that in every irregular pair V_i, V_j, we find $S \subseteq V_i$, $T \subseteq V_j$, $|S|, |T| \geq \epsilon\nu$, $\nu \geq \lfloor (1-\epsilon)n/k \rfloor$ such that $\Delta(S,T) \geq \epsilon^3 \nu^2/8$. Subtraction of the corresponding cut matrices reduces the Frobenius norm by at least $3\epsilon^6\nu^2/192$. Also, the non-zero parts of the different cut matrices are disjoint from each other. Inequality (ii) follows from Lemma 7(b). Inequality (iii) follows from the fact that each V_i is cut into at most 4^k pieces by this construction.

$\mathcal{Q}$ may not be equitable and so as in Section 5.1.1 we split each W_i into sets $W_{i,j}$ of size $\mu = \lfloor \epsilon n/(k\ell) \rfloor$ and add a total $\leq \epsilon n/k$ elements to the remainder set. Consider a new partition $\mathcal{P}'$ consisting of the sets $W_{i,j}$ plus the remainder set. It follows from Lemma 7(b) that

$$||\mathbf{A} \circ \mathcal{P}'||_F \leq ||\mathbf{A} \circ \mathcal{Q}||_F,$$

and (51) follows. □

6 Multidimensional Extensions

In this section, we consider higher dimensional matrices. We apply our decompositions recursively. Only the notation presents any real difficulty. Consequently we will be content to sketch the proofs. Suppose $r \geq 3$ and $X_1, X_2, \ldots X_r$ are finite sets. An r-dimensional matrix $\mathbf{M}$ *on* $X_1 \times X_2 \times \ldots X_r$ is a map

$$\mathbf{M}: \; X_1 \times X_2 \cdots \times X_r \rightarrow \mathbf{R}.$$

If $S_i \subseteq X_i$ for $i = 1, 2, \ldots r$, and d is a real number the matrix $\mathbf{M}$ satisfying

$$\mathbf{M}(e) = \begin{cases} d & \text{for } e \in S_1 \times S_2 \cdots \times S_r \\ 0 & \text{otherwise} \end{cases}$$

is called a cut matrix and is denoted

$$\mathbf{M} = CUT(S_1, S_2, \ldots S_r; d).$$

We will show that we can usefully approximate any $r-$dimensional matrix as the sum of a small number of cut matrices.

We need to extend some of the notation from 2-dimensional matrices to r-dimensional ones. For $S_i \subseteq X_i$, $i = 1, 2, \ldots r$, we define

$$\mathbf{M}(S_1, S_2, \ldots, S_r) = \sum_{e \in S_1 \times S_2 \times \cdots \times S_r} \mathbf{M}(e).$$

We then let

$$\begin{aligned} ||\mathbf{M}||_C &= \max\{|\mathbf{A}(S_1, S_2, \ldots, S_r)| : \; S_i \subseteq X_i \text{ for } i = 1, 2, \ldots, r\}. \\ ||\mathbf{M}||_F &= \left(\sum_{e \in X_1 \times X_2 \times \cdots \times X_r} \mathbf{A}(e)^2 \right)^{1/2} \end{aligned}$$

A cut decomposition of an r-dimensional matrix $\mathbf{A}$ has the same form as before, (2). The notions of width, coefficient length and error are defined as in the 2-dimensional case.

We wish to extend both of our decomposition algorithms. Let $\Delta = \prod_{i=1}^r |X_i|$. In the case of the First Algorithm we assume $||\mathbf{A}||_\infty \leq 1$ and define $\rho = \Delta$. For the Second Algorithm we define $\rho = \Delta^{1/2} ||\mathbf{A}||_F$.

Theorem 11 *Suppose $\mathbf{A}$ is an $r-$dimensional matrix on $X_1 \times X_2 \times \cdots \times X_r$. We assume that $r \geq 3$ is fixed. Suppose ϵ, δ are reals in $[0,1]$. We can with probability at least $1 - \delta$ find a cut decomposition of error at most $\epsilon\rho$. Either (First Algorithm) the width is $O(\epsilon^{4-4r})$, the running time is $O(r^{O(1)} \epsilon^{-O(\log_2 r)} \delta^{-1})$ and the coefficient length is at most C^r for some absolute constant $C > 0$, or (Second Algorithm) the width is $O(\epsilon^{2-2r})$, the running time is $O(r^{O(1)} \epsilon^{-O(\log_2 r)} 2^{\tilde{O}(1/\epsilon^2)} \delta^{-2})$ and the coefficient length is at most $C^r ||\mathbf{A}||_F^2 / \Delta$.*

Proof Let $\mathbf{B}$ be the following (2-dimensional) matrix with rows indexed by $Y_1 = X_1 \times \cdots \times X_{\hat{r}}$, $\hat{r} = \lfloor r/2 \rfloor$ and columns indexed by $Y_2 = X_{\hat{r}+1} \times \cdots \times X_r$. If $i = (x_1, \ldots, x_{\hat{r}}) \in Y_1$ and $j = (x_{\hat{r}+1}, \ldots, x_r) \in Y_2$ then $\mathbf{B}(i,j) = \mathbf{A}(x_1, x_2, \ldots, x_r)$. Applying a decomposition algorithm we obtain

$$\mathbf{B} = \mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \cdots + \mathbf{D}^{(s_0)} + \mathbf{W}$$

where for $1 \leq t \leq s_0$,

$$\mathbf{D}^{(t)} = CUT(R_t, C_t, d_t),$$

$$||\mathbf{W}||_C \leq \epsilon\rho/2 \text{ and } \sum_{t=1}^{s_0} d_t^2 \leq 27||\mathbf{A}||_F^2/\Delta. \tag{53}$$

Each R_t defines an $\hat{r}$-dimensional 0-1 matrix $\mathbf{R}^{(t)}$ where $\mathbf{R}^{(t)}(x_1, \ldots, x_{\hat{r}}) = 1$ iff $(x_1, \ldots, x_{\hat{r}}) \in R_t$. $\mathbf{C}^{(t)}$ is defined similarly. Assume inductively that we can further decompose

$$\begin{aligned}
\mathbf{R}^{(t)} &= \mathbf{D}^{(t,1)} + \cdots + \mathbf{D}^{(t,s_1)} + \mathbf{W}^{(t)} \\
\mathbf{C}^{(t)} &= \hat{\mathbf{D}}^{(t,1)} + \cdots + \hat{\mathbf{D}}^{(t,\hat{s}_1)} + \hat{\mathbf{W}}^{(t)}
\end{aligned}$$

Here

$$\begin{aligned}
\mathbf{D}^{(t,u)} &= CUT(R_{t,u,1}, \ldots, R_{t,u,\hat{r}}, d_{t,u}) \qquad 1 \leq t, u \leq s_1, \\
\hat{\mathbf{D}}^{(t,\hat{u})} &= CUT(R_{t,\hat{u},\hat{r}+1}, \ldots, R_{t,\hat{u},r}, \hat{d}_{t,\hat{u}}) \qquad 1 \leq t, \hat{u} \leq \hat{s}_1,
\end{aligned}$$

where

$$\begin{aligned}
R_{t,u,i} &\subseteq X_i \qquad 1 \leq i \leq \hat{r}, \\
\hat{R}_{t,\hat{u},i} &\subseteq X_{\hat{r}+i} \qquad 1 \leq i \leq r - \hat{r},
\end{aligned}$$

$$||\mathbf{W}^{(t)}||_C \leq \epsilon_2 \prod_{i=1}^{\hat{r}} |X_i| \text{ and } ||\hat{\mathbf{W}}^{(t)}||_C \leq \epsilon_2 \prod_{i=\hat{r}+1}^{r} |X_i|,$$

and $\epsilon_2 = \epsilon/(K s_0^{1/2})$ for some suitably large constant $K > 0$.

If $\hat{r} = 1$ then $\mathbf{R}^{(t)}$ is a 0-1 X_1-vector and this is also a 1-dimensional cut matrix.

It follows that we can write

$$\mathbf{A} = \sum_{t=1}^{s_0} \sum_{u=1}^{s_1} \sum_{\hat{u}=1}^{\hat{s}_1} CUT(R_{t,u,1}, \ldots, R_{t,u,\hat{r}}, \hat{R}_{t,\hat{u},\hat{r}+1}, \ldots, \hat{R}_{t,\hat{u},r}, d_{t,u,\hat{u}}) + \mathbf{W}_1.$$

Here, for $1 \leq t \leq s_0, 1 \leq u \leq s_1, 1 \leq \hat{u} \leq \hat{s}_1$,

$$d_{t,u,\hat{u}} = d_t d_{t,u} \hat{d}_{t,\hat{u}}$$

and for $S = T_1 \times T_2$, $T_1 = S_1 \times \cdots \times S_{\hat{r}}$, $T_2 = S_{\hat{r}+1} \times \cdots \times S_r$,

$$\mathbf{W}_1(S) = \mathbf{W}(S) + \sum_{t=1}^{s_0} d_t(\mathbf{W}^{(t)}(T_1)\hat{\mathbf{W}}^{(t)}(T_2) + \mathbf{W}^{(t)}(T_1)\mathbf{C}^{(t)}(T_2) + \mathbf{R}^{(t)}(T_1)\hat{\mathbf{W}}^{(t)}(T_2)).$$

Hence

$$\begin{aligned}
||\mathbf{W}_1||_C &\leq ||\mathbf{W}||_C + 3\epsilon_2\Delta \sum_{t=1}^{s_0} |d_t| \\
&\leq ||\mathbf{W}||_C + 3\epsilon_2\Delta s_0^{1/2} \left(\sum_{t=1}^{s_0} d_t^2\right)^{1/2}.
\end{aligned}$$

In the case of the Second Algorithm we see that our bound on the coefficient length of the decomposition implies

$$\begin{aligned}
||\mathbf{W}_1||_C &\leq ||\mathbf{W}||_C + 18\epsilon_2 s_0^{1/2}\rho \\
&\leq \epsilon\rho/2 + 18\epsilon_2 s_0^{1/2}\rho \\
&\leq \epsilon\rho.
\end{aligned} \tag{54}$$

We see that the coefficient length, (for the second algorithm), of our decomposition is

$$\sum_{t=1}^{s_0} d_t^2 \left(\sum_{u=1}^{s_1} d_{t,u}^2\right)^2 \left(\sum_{\hat{u}=1}^{\hat{s}_1} d_{t,\hat{u}}^2\right)^2.$$

Assume inductively, that for some $L > 0$ we have

$$\sum_{u=1}^{s_1} d_{t,u}^2 \leq L^{2\hat{r}-1} \text{ and } \sum_{\hat{u}=1}^{\hat{s}_1} d_{t,\hat{u}}^2 \leq L^{2(r-\hat{r})-1}.$$

Then the coefficient length of the decomposition is at most

$$L^{2r-2}\sum_{t=1}^{s_0} d_t^2 \leq L^{2r-2}\frac{27||\mathbf{A}||_F^2}{\Delta},$$

(See (53). Putting $C = \sqrt{27}$ yields our bound on the coefficient length for the second algorithm.

The analysis for the First Algorithm is similar.

Note finally that the claimed running times and sizes of the partitions can be verified by induction. □

6.1 Hypergraph Partitions

We note next that the the matrix decomposition described above can be used to partition hypergraphs as we did for graphs in Section 5. See Chung [10], Frankl and Rödl [14] and Prömel and Steger [25] for non-constructive versions of Szemerédi's lemma in hypergraphs.

Let $H = (V, E)$ be an r-uniform hypergraph, i.e. each $e \in E$ is of size r. For disjoint sets $A_1, A_2, \dots, A_r$ we let $e(A_1, A_2, \dots, A_r)$ denote the number of edges $e = \{v_1, v_2, \dots, v_r\}$ such that $v_i \in A_i$, $1 \leq i \leq r$. The density

$$d(A_1, A_2, \dots, A_r) = \frac{e(A_1, A_2, \dots, A_r)}{|A_1||A_2|\cdots|A_r|}.$$

A partition $\mathcal{P} = V_1, V_2, \dots, V_k$ of V is said to be ϵ-pseudo-regular if for all disjoint sets $S_1, S_2, \dots, S_r \subseteq V$ we have

$$\left|e(S_1, S_2, \dots, S_r) - \sum_{i_1, i_2, \dots, i_r} d(V_{i_1}, V_{i_2}, \dots, V_{i_r}) \prod_{t=1}^{r} |V_{i_t} \cap S_t|\right| \leq \epsilon n^r.$$

This notion generalises what we have already seen in Section 5.1 for the case $r = 2$. Assuming r is fixed, an ϵ-pseudo-regular partition can be computed using Theorem 11 in an analogous manner to that used for the case $r = 2$.

Similarly, we can compute an ϵ-RL partition generalising the results of Section 5.2. Here, given ϵ, m we compute a partition $\mathcal{P} = V_1, V_2, \dots, V_k$ of V such that

- $m \leq k \leq Q_r(\epsilon, m)$.
- $|V_0| \leq \epsilon n^r$.
- $|V_1| = |V_2| = \cdots = |V_k|$.

- All but ϵk^r of the r-tples $(V_{i_1}, V_{i_2} \ldots, V_{i_r})$ are ϵ-regular.

Here $(V_{i_1}, V_{i_2} \ldots, V_{i_r})$ is ϵ-regular if for all $A_{i_t} \subseteq V_{i_t}$, $|A_{i_t}| \geq \epsilon |V_{i_t}|$, $1 \leq t \leq r$,

$$|d(A_{i_1}, A_{i_2} \ldots, A_{i_r}) - d(V_{i_1}, V_{i_2} \ldots, V_{i_r})| \leq \epsilon.$$

Both of the above results follow from the following two lemmas. The first is a generalisation of ideas of Section 4.3. The adjacency matrix $\mathbf{A}$ of H is the $V \times V \times \cdots \times V$ 0-1 matrix with $\mathbf{A}(v_1, v_2, \ldots, v_r) = 1$ iff $(v_1, v_2, \ldots, v_r) \in E$.

Lemma 8 *Suppose* $\mathbf{W}$ *is an* $r-$*dimensional matrix on* $R_1 \times R_2 \times \ldots R_r$. *Suppose* ϵ, δ *are reals in* $[0,1]$. *Then there is a polymomial time algorithm which returns either the statement that*

$$||\mathbf{W}||_C \leq \epsilon\rho ||\mathbf{A}||_F,$$

or returns sets $Z_i \subseteq R_i$, $i = 1, 2, \ldots, r$ *such that*

$$|\mathbf{W}(Z_1, Z_2, \ldots Z_r)| \geq \epsilon\rho |\mathbf{A}||_F/2. \tag{55}$$

The algorithm is correct with probability at least $1 - \delta$.

Proof Suppose that for some $S_1 \subseteq R_1, S_2 \subseteq R_2, \ldots S_r \subseteq R_r$, we have

$$|\mathbf{W}(S_1, S_2, \ldots S_r)| \geq \epsilon\rho ||\mathbf{A}||_F.$$

Assume without loss of generality that in fact

$$\mathbf{W}(S_1, S_2, \ldots S_r) \geq \epsilon\rho ||\mathbf{A}||_F.$$

$Z_1, Z_2, \ldots Z_r$ will be constructed by a process and if

$$\mathcal{E}_i = \{\mathbf{W}(Z_1, Z_2, \ldots, Z_i, S_{i+1}, \ldots, S_r) \geq \epsilon\rho ||\mathbf{A}||_F (1 - \tfrac{i}{2r})\}$$

then it will be sufficient to prove that

$$\mathbf{Pr}(\mathcal{E}_i \mid \mathcal{E}_{i-1}) \geq 1 - \tfrac{\delta}{r}. \tag{56}$$

This is because $\mathcal{E}_0$ is our assumption and $\mathcal{E}_r$ is our conclusion. So fix i and assume $\mathcal{E}_{i-1}$ occurs. Let $p = 4r^2\epsilon^{-2}$ and let X be a random p-subset of $L_i = Z_1 \times \cdots \times Z_{i-1} \times R_{i+1} \times \cdots \times R_r$. For each $x \in X$ and $a \in R_i$ define $a \star x \in R_1 \times \cdots \times R_r$ by inserting a as the ith component. For $Y \subseteq X$ let $a \star Y = \{a \star x : \ x \in Y\}$ and $A_i(Y) = \{a \in R_i : \ W(a \star Y) \geq 0\}$. Let $Y_0 = X \cap (Z_1 \times \cdots \times Z_{i-1} \times S_{i+1} \times \cdots \times S_r)$.

Taking $Z_i = A_i(Y_0)$ we see that equation (56) follows from

$$\mathbf{Pr}(\mathbf{W}(Z_1, Z_2, \ldots, Z_{i-1}, A_i(Y_0), S_{i+1}, \ldots, S_r) \leq \epsilon\rho ||\mathbf{A}||_F (1 - \tfrac{1}{2r}) \mid \mathcal{E}_{i-1}) \leq \tfrac{\delta}{r},$$

and this follows by the Markov inequality from

$$\begin{aligned}
&\mathbf{E}(\mathbf{W}(Z_1, Z_2, \ldots, Z_{i-1}, A_i(Y_0), S_{i+1}, \ldots, S_r)) \geq \\
&\quad \mathbf{W}(Z_1, Z_2, \ldots, Z_{i-1}, P_{\mathbf{W}}(Z_1, Z_2, \ldots, Z_{i-1}, *, S_{i+1}, \ldots, S_r), S_{i+1}, \ldots, S_r) - \tfrac{\rho}{\sqrt{p}} ||\mathbf{W}||_F \geq \\
&\qquad \mathbf{W}(Z_1, Z_2, \ldots, Z_{i-1}, S_i, S_{i+1}, \ldots, S_r), S_{i+1}, \ldots, S_r) - \tfrac{\rho}{\sqrt{p}} ||\mathbf{W}||_F.
\end{aligned} \tag{57}$$

Here

$$P_{\mathbf{W}}(Z_1, Z_2, \ldots, Z_{i-1}, *, S_{i+1}, \ldots, S_r) = \{a \in R_i : \ \mathbf{W}(Z_1, Z_2, \ldots, Z_{i-1}, a, S_{i+1}, \ldots, S_r) \geq 0\}.$$

Equation (57) follows from Lemma 6. We take $R = L_i, C = R_i$ and $Y = Z_1 \times \cdots \times Z_{i-1} \times S_{i+1} \times \cdots \times S_r$.

Our algorithm for computing $Z_1, Z_2, \ldots, Z_r$ is then:

begin
  Choose X_1 randomly from L_1.
    For each $Y_1 \subseteq X_1$ let $Z_1 = A_1(Y_1)$.
      Choose X_2 randomly from L_2.
        For each $Y_2 \subseteq X_2$ let $Z_2 = A_2(Y_2)$.
$$\vdots$$
end

Then output the choice of $Z_i = A_i(Y_i)$ which maximises $\mathbf{W}(Z_1, Z_2, \ldots, Z_r)$ over the 2^{rp} choices. □

In this proof we have avoided the probe model in order to make the proof less technical. It should be clear that by doing more sampling we can make a constant time algorithm.

Lemma 9 *If, in the notation of Lemma 8, (55) holds and* $\mathbf{D} = Cut(Z_1, Z_2, \ldots, Z_r, d)$ *where* $d = \mathbf{W}(Z_1, Z_2, \ldots, Z_r)/(|Z_1|\,|Z_2| \cdots |Z_r|)$ *then*

$$||\mathbf{W}||_F^2 - ||\mathbf{W} - \mathbf{D}||_F^2 \geq \frac{\epsilon^2 ||\mathbf{A}||_F^2}{4}.$$

To find an 2ϵ-pseudo-regular parttion of H we use Lemmas 8 and 9 to construct $s \leq 4\epsilon^{-2}$ cut matrices $\mathbf{D}^{(i)} = Cut(Z_1^{(i)}, Z_2^{(i)}, \ldots, Z_s^{(i)}, d_i)$ such that $||\mathbf{A} - (\mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \cdots + \mathbf{D}^{(s)})||_C \leq \epsilon n^r$. We then compute the coarsest partition $\mathcal{P} = V_1, V_2, \ldots, V_k$ which is a simultaneous refinement of $Z_j^{(i)}$, $1 \leq i \leq s$, $1 \leq r$. We then prove that it is 2ϵ-pseudo-regular in an analogous manner to (50). We can make it equitable as in Section 5.1.1.

To find an ϵ-RL partition we follow the same approach as for the case $r = 2$. For a current partition $\mathcal{P} = V_0, V_1, \ldots, V_k$ we consider the matrix $\mathbf{A} \circ \mathcal{P}$ (defined analogously as for the case $r = 2$) and its collection of $V_{i_1} \times V_{i_2} \times \cdots \times V_{i_r}$ submatrices $\mathbf{A}_{i_1, i_2, \ldots, i_r}$. If the tple $V_{i_1}, V_{i_2}, \ldots, V_{i_r}$ is not ϵ-regular then $||\mathbf{A}_{i_1, i_2, \ldots, i_r}||_C \geq \epsilon^{r+1} \lfloor (1-\epsilon)n/k \rfloor^r$ and then by Lemmas 8 and 9 we can construct a cut matrix $\mathbf{D}_{i_1, i_2, \ldots, i_r}$ which is zero outside $V_{i_1} \times V_{i_2} \times \cdots \times V_{i_r}$ and is such that

$$||\mathbf{A} \circ \mathcal{P} - \mathbf{D}_{i_1, i_2, \ldots, i_r}||_F^2 \leq ||\mathbf{A} \circ \mathcal{P}||_F^2 - \frac{\epsilon^{2r+2} n^r}{5k^r},$$

for ϵ small.

So if $\mathcal{P}$ is not ϵ-RL we can find a sum of cut matrices $\mathbf{D}$ such that

$$||\mathbf{A} \circ \mathcal{P} - \mathbf{D}||_F^2 \leq ||\mathbf{A} \circ \mathcal{P}||_F^2 - \frac{\epsilon^{2r+3} n^r}{5}. \tag{58}$$

We then, as in the case $r = 2$, find the coarsest refinement of $\mathcal{P}$ which is also a refinement of each of the sets used in each of the cut matrices. We finalll make it equitable and produce $\mathcal{P}'$. The proof given for $r = 2$ can be generalised with only minor changes, since Lemma 7 can be extended to general r. Then (58) implies that the process stops after at most $5\epsilon^{-(2r+3)}$ iterations.

7 Max-SNP Problems

Let MAX-r-FUNCTION-SAT be the problem where the input consists of m Boolean functions $f_1, f_2, \ldots f_m$ in n variables - $V = \{u_1, u_2, \ldots u_n\}$, but where each f_i depends on only r variables

(r fixed). The aim is to assign truth values to the n variables, so as to satisfy as many of the f_i as possible. It is well-known [23] that a Max-SNP problem can be viewed as a MAX-r-FUNCTION-SAT problem for a fixed r.

We may formulate the MAX-r-FUNCTION-SAT problem as follows: There are at most $\ell = 2^{2^r}$ possible Boolean functions of r variables; we number them $1, 2, \ldots \ell$. We will have ℓ r-dimensional matrices $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \ldots \mathbf{A}^{(\ell)}$ on $V \times V \times \cdots \times V$, with 0-1 entries to represent the data of the problem. The matrix $\mathbf{A}^{(p)}$ will have a 1 in the $(i_1, i_2, \ldots i_r)$ entry iff there is an f_i among the given functions $f_1, f_2, \ldots f_m$ which has as its arguments $u_{i_1}, u_{i_2}, \ldots u_{i_r}$ and is the pth function of these arguments; for convenience, we then say that the *type* of this function f_i is p.

Suppose for the moment we have in mind a fixed truth assignment $T : V \to \{0, 1\}$. We will also denote by T the set $\{u : T(u) = 1\}$.

We may express each function in Disjunctive Normal Form. So based only on the type p of a function f_i, we can determine a subset Q_p of $\{0, 1\}^r$ such that $f_i(u_{i,1}, u_{i,2}, \ldots u_{i,r})$ is TRUE under T iff

$$(T(u_{i,1}), T(u_{i,2}), \ldots T(u_{i,r})) \in Q_p.$$

For each r-tuple of variables, $(u_{i_1}, u_{i_2}, \ldots u_{i_r}) = e$ (say), we let $T(e)$ denote the r-tuple $(T(u_{i_1}), T(u_{i_2}), \ldots T(u_{i_r}))$. Then we have

$$|\{i : f_i = 1 \text{ under } T\}| = \sum_p \sum_{a \in Q_p} |\{e : \mathbf{A}^{(p)}(e) = 1;\ T(e) = a\}|. \tag{59}$$

For $a \in \{0, 1\}^r$, and $1 \leq q \leq r$, define

$$S_q(a) = \{v \in V :\ T(v) = a_q\}.$$

Let

$$S(a) = S_1(a) \times S_2(a) \times \cdots \times S_r(a).$$

Then,

$$\sum_p \sum_{a \in Q_p} |\{e : \mathbf{A}^{(p)}(e) = 1;\ T(e) = a\}| = \sum_p \sum_{a \in Q_p} \mathbf{A}^{(p)}(S(a)). \tag{60}$$

We will approximately maximize the right hand side of (60) and so approximately maximise our actual objective, (59).

To this end, we find r-dimensional Cut matrices $\{\mathbf{D}_t^{(p)} :\ p = 1, 2, \ldots, \ell; t = 1, 2, \ldots s\}$ where $s = O\epsilon^{2-2r})$ and such that

$$||\mathbf{A}^{(p)} - (\mathbf{D}_1^{(p)} + \mathbf{D}_2^{(p)} + \cdots + \mathbf{D}_s^{(p)})||_C \leq \epsilon n^r / (8 \times 2^{2^r} 2^r), \tag{61}$$

where $\mathbf{D}_t^{(p)} = Cut(R_{t,1}^{(p)}, R_{t,2}^{(p)}, \ldots, R_{t,r}^{(p)}, d_t^{(p)})$.

Now,

$$\sum_p \sum_{a \in Q_p} \mathbf{A}^{(p)}(S(a)) = \sum_p \sum_{a \in Q_p} \sum_{t=1}^{s} \mathbf{D}_t^{(p)}(S(a)) + \Delta_1,$$

where

$$|\Delta_1| \leq 2^{2^r} 2^r \epsilon n^r / (8 2^{2^r} 2^r) = \epsilon n^r / 8.$$

Now,

$$\sum_p \sum_{a \in Q_p} \sum_{t=1}^{s} \mathbf{D}_t^{(p)}(S(a)) = \sum_p \sum_{a \in Q_p} \sum_{t=1}^{s} d_t^{(p)} \prod_{q=1}^{r} |S_q(a) \cap R_{t,q}^{(p)}|.$$

Let

$$|R_{t,q}^{(p)} \cap S_q(a)| = f_{t,q}^{(p)}(a) \qquad \text{for } 1 \le t \le s\ ;\ 1 \le p \le \ell\ ;\ 1 \le q \le r\ ;\ a \in \{0,1\}^r.$$

Let $K = C^r$, our bound on the coefficient length of the decomposition, so that $|d_t^{(p)}| \le K$. Let $\nu = \epsilon n/(8Ks\ell 4^r)$ and

$$g_{t,q}^{(p)}(a) = \left\lfloor \frac{f_{t,q}^{(p)}(a)}{\nu} \right\rfloor \nu. \tag{62}$$

Note that $f_{t,q}^{(p)}(a) \le n$, and so, for each t, p, a,

$$|\prod_{q=1}^{r} f_{t,q}^{(p)}(a) - \prod_{q=1}^{r} g_{t,q}^{(p)}(a)| \le n^{r-1} 2^r \nu.$$

Then we have that

$$\sum_p \sum_{a \in Q_p} \sum_{t=1}^{s} d_t^{(p)} \prod_{q=1}^{r} |S_q(a) \cap R_{t,q}^{(p)}| = \Delta_2 + \sum_p \sum_{a \in Q_p} \sum_{t=1}^{s} d_t^{(p)} \prod_{q=1}^{r} g_{t,q}^{(p)}, \tag{63}$$

where $|\Delta_2| \le s\ell 4^r n^{r-1} \nu K \le \epsilon n^r/8$.

Thus the number of functions f_i satisfied by our assignment T is almost determined by the values $g_{t,q}^{(p)}(a)$. We consider how to find the "best" set of values.

Now, each $g_{t,q}^{(p)}(a)$ has $O(\epsilon^{1-2r})$ possible values, so the total number of sets of values for all $g_{t,q}^{(p)}(a)$ is $O((1/\epsilon^{1-2r})^{O(\epsilon^{2-2r})})$, ($r$ is constant).

As in previous algorithms we enumerate all these sets of values. We argue that for each set of such values, we can check (approximately) by a linear program in $O((\epsilon^{1-2r})^{O(\epsilon^{2-2r})})$ variables if there is some set of feasible $f_{t,q}^{(p)}(a)$ (feasible means that these values can be attained by for some truth assignment T) whose "round down" is the enumerated $g_{t,q}^{(p)}(a)$. To this end, let $\mathcal{P}$ be the coarsest partition of V (with at most $2^{s\ell}$ parts in it) such that each $R_{t,q}^{(p)}$ is the union of some sets in $\mathcal{P}$. We explicitly construct $\mathcal{P}$. For each $P \in \mathcal{P}$, let $x_P = |T \cap P|$; these are to be determined.

It is easy to see that all the $f_{t,q}^{(p)}(a)$ can be expressed as sums of these x_P. So, given a set of values of $g_{t,q}^{(p)}(a)$, we may write the following Integer Program with variables x_P. (Note that if the $g_{t,q}^{(p)}(a)$ arise from some assignment T via (62) then $\{x_P = |P \cap T|\}$ will be feasible):

$$\begin{array}{lllll} 0 & \le & x_P & \le & |P| \qquad \forall P \in \mathcal{P} \\ g_{t,q}^{(p)}(a) & \le & \displaystyle\sum_{P \subseteq R_{t,q}^{(p)}} x_P & \le & g_{t,q}^{(p)}(a) + \nu \quad \text{for } t,p,q,a \text{ with } a_q = 1 \\ g_{t,q}^{(p)}(a) & \le & \displaystyle\sum_{P \subseteq R_{t,q}^{(p)}} (|P| - x_P) & \le & g_{t,q}^{(p)}(a) + \nu \quad \text{for } t,p,q,a \text{ with } a_q = 0. \end{array}$$

Consider the Linear Programming relaxation of this Integer Program. There are two possibilities: (a) it is infeasible in which case the Integer Program is also infeasible; (b) there is a feasible solution x_P to the Linear Program. We round down each x_P to the nearest integer (below it) to get y_P. Then, we have for each t, p, q, a, with $S_q(a) = T$, the upper bound on $\sum_{P \subseteq R_{t,q}^{(p)}} x_P$ is still satisfied; and also we have

$$\sum_{P \subseteq R_{t,q}^{(p)}} y_P \ge g_{t,q}^{(p)}(a) - 2^{sl}.$$

Similary for t, p, q, a with $S_q(a) = V \setminus T$, we have

$$\sum_{P \subseteq R_{t,q}^{(p)}} (|P| - y_P) \le g_{t,q}^{(p)}(a) + \nu + 2^{sl}.$$

So for any T^* with $|T^* \cap P| = y_P$ for all $P \in \mathcal{P}$ (such T^*'s obviously exist since $\mathcal{P}$ is a partition), we have that

$$||R_{t,q}^{(p)} \cap S_q(a)| - g_{t,q}^{(p)}(a)| \le \nu + 2^{sl} \le 2\nu,$$

for n high enough, since $2^{sl} = 2^{\tilde{O}(\epsilon^{2-2r})}$.

This implies that (arguing as in (63), for each feasible set of $g_{t,q}^{(p)}(a)$, we find a T^* with the difference between the actual number of functions satisfied by T^* and the approximate value given by $g_{t,q}^{(p)}(a)$ is at most $\epsilon n^r/2$, so it suffices to compute the best $g_{t,q}^{(p)}(a)$ among the enumerated ones which is found to be (approximately) feasible by the above.

8 Continuous Case

We finally give an existence result where m and n are infinite. Let $f : [0,1]^2 \to \mathbf{R}$ be a (Lebesgue) measurable function and assume that

$$||f||_2^2 = \int_{[0,1]^2} f(x,y)^2 dxdy < \infty.$$

For measurable $S, T \subseteq [0,1]$ we let

$$f(S,T) = \int_{S \times T} f(x,y) dxdy.$$

Then define

$$||f||_C = \sup_{S,T} |f(S,T)|.$$

A function g is a *cut function* if there exist measurable S, T and real d such that

$$g(x,y) = \begin{cases} d & (x,y) \in S \times T, \\ 0 & \text{otherwise.} \end{cases}$$

We will use the notation $g = CUT(S,T,d)$.

Theorem 12 *There exist cut functions $f_1, f_2, \ldots, f_s$, $s \le 1/\epsilon^2$ such that if*

$$w_t = f - (f_1 + f_2 + \cdots + f_t)$$

then

$$||w_s||_C \le \epsilon ||f||_2. \tag{64}$$

Proof Assume inductively that we have found cut functions

$$f_j = CUT(S_j, T_j, d_j) \quad 0 \le j \le t \quad (f_0 = 0),$$

such that

$$||w_t||_2^2 \le (1 - \epsilon^2 t)||f||_2^2.$$

Either (64) holds (with $s = t$) or there exist $S, T \subseteq [0, 1]$ such that $|w_t(S, T)| > \epsilon ||f||_2$. Let $S_{t+1} = S, T_{t+1} = T$ and $d = d_{t+1} = w_t(S, T)/(|S||T|)$. ($|S|$ denotes the measure of S). Then

$$
\begin{aligned}
||w_{t+1}||_2^2 - ||w_t||_2^2 &= \int_{S\times T} ((w_t(x,y) - d)^2 - w_t(x,y)^2) dxdy \\
&= -|S|\,|T| d^2 \\
&= -\frac{w_t(S,T)^2}{|S|\,|T|} \\
&\leq -\epsilon^2 ||f||_2^2.
\end{aligned}
$$

The theorem follows. □

Final Remark: As a referee has pointed out, the approximation algorithms can easily be derandomised. The matrix decomposition requires a constant number of probes and for each probe there are a polynomial number of choices and so the space of choices is polynomial.

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