Robust Secret Sharing Schemes Against Local Adversaries

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Secret Sharing (Informal)

(Share, Rec) pair of algorithms:

$$s \xmapsto{\text{Share}} (s_1, \ldots, s_n) \xmapsto{\text{Rec}} s$$

***t*-privacy:** $s_1, \ldots, s_t \quad \Rightarrow \quad$ no info on s

***r*-reconstructability:** $s_1, \ldots, s_r \quad \Rightarrow \quad s$ uniquely determined

For **threshold schemes**: $r = t + 1$.

Example: Shamir Secret Sharing [Sha79]

$\mathbb{F}$ field, public $x_1, \ldots, x_n \in \mathbb{F}$.

Shamir.Share$_t(s)$:

1. pick uniform $a_1, \ldots, a_t \in \mathbb{F}$
2. define polynomial $f(X) := s + \sum_{j=1}^{t} a_j X^j \in \mathbb{F}[X]$
3. compute $s_i \leftarrow f(x_i)$
4. output $(s_1, \ldots, s_n)$

Shamir.Rec$_t(s_1, \ldots, s_n)$:

1. Lagrange interpolation to recover $f(X)$
2. output $f(0)$

Robust Secret Sharing – Standard Model

(Share, Rec) Secret Sharing, (t, δ)-**robust**: for any Adv,

$$\begin{tikzcd}
s \arrow[r, mapsto, "\text{Share}"] & (s_1, \ldots, s_t, s_{t+1}, \ldots, s_n) \arrow[d, mapsto, "(\widetilde{s}_1,\ldots,\widetilde{s}_t)=\text{Adv}(s_1,\ldots,s_t)"] & \\
& (\widetilde{s}_1, \ldots, \widetilde{s}_t, s_{t+1}, \ldots, s_n) \arrow[r, mapsto, "\text{Rec}"] & s'
\end{tikzcd}$$

$$\Pr[s' \neq s] \leq \delta$$

Robust Secret Sharing – with Local Adversaries

(Share, Rec) Secret Sharing, **locally** (t, δ)-**robust**: for any $\mathsf{Adv}_1, \ldots, \mathsf{Adv}_t$,

$$
\begin{tikzcd}
s \arrow[r, mapsto, "\mathsf{Share}"] & (s_1, \ldots, s_t, s_{t+1}, \ldots, s_n) \arrow[d, mapsto, "\widetilde{s_1}=\mathsf{Adv}_1(s_1),\ldots,\widetilde{s_t}=\mathsf{Adv}_t(s_t)"] & \\
& (\widetilde{s_1}, \ldots, \widetilde{s_t}, s_{t+1}, \ldots, s_n) \arrow[r, mapsto, "\mathsf{Rec}"] & s'
\end{tikzcd}
$$

$$\Pr[s' \neq s] \leq \delta$$

Does Locality Make Sense?

It captures the following:

Pre-Game: Players talk to each other, set their actions

Game:
- Players are given private inputs
- Players run protocol without revealing inputs to others
- Output of protocol is set

Post-Game: Players talk to each other again, possibly revealing inputs

Similar to collusion-free protocols [LMs05].

Locality – Possible Scenarios

- Corrupt parties unwilling to coordinate (e.g. different goals)
- Corrupt parties oblivious about existence of each other
- Network with (independently) faulty channels
- Data is required to travel fast, coordination impossible
- ...

Locality – Related Work

Interactive Proofs:

- Multi-prover interactive proofs:
  MIP=NEXP, [BFL91] (IP=PSPACE, [Sha92])

Multi-party Computation:

- Collusion-free protocols [LMs05, AKL$^+$09, AKMZ12]
- Local UC [CV12]

Leakage-resilient crypto:

- Split secret state and independent leakage [DP08]

Facts about Robust Secret Sharing

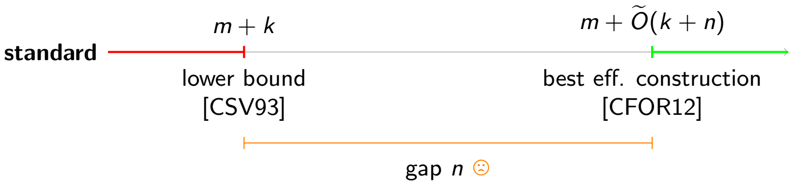
$t < n/3$: perfect robustness ($\delta = 0$)
no share size overhead ($|s_i| = |s| =: m$)
e.g. Shamir share + Reed-Solomon decoding
RS decodes up to $(n - t)/2 > (3 \cdot t - t)/2 = t$ errors

$n/3 \leq t < n/2$: tricky!
no perfect robustness ($\delta = 2^{-k}$) [Cev11]
shares larger than secret ($|s_i| > m$) [Cev11]

All of the above: independent of standard/local adv. model

The Tricky Case

The trickiest case: $n = 2 \cdot t + 1$.
Analysis of $|s_i|$:



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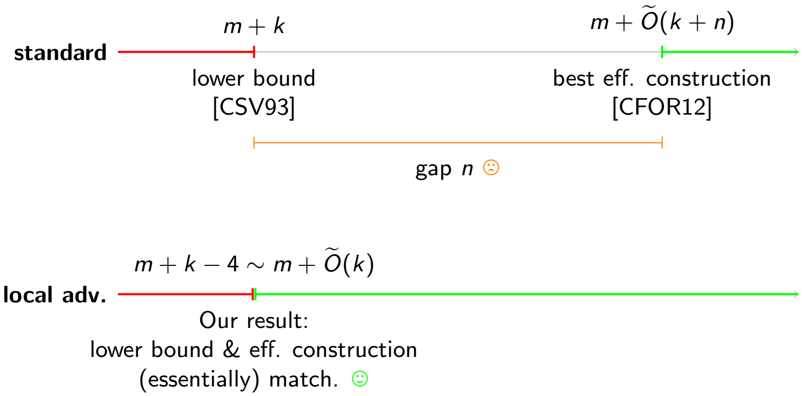

Our Construction[1]

Previous Constructions

Privacy: Shamir secret sharing, degree$=t$

Robustness: one-time MAC, $O(n)$ keys per player.

$\Rightarrow |s_i|$ inherent depends (at least) linearly on n

Our Construction

Privacy: Shamir secret sharing, degree$=t$

Robustness: one-time MAC, one key only.

[1] Conceptually simpler; thanks to Daniel Wichs for fruitful discussions.

In Detail

Share(s):

1. sample MAC key $z \in X$
2. $(s_1, \dots, s_n) \leftarrow \text{Shamir.Share}_t(s)$
3. $(z_1, \dots, z_n) \leftarrow \text{Shamir.Share}_1(z)$
4. $t_i \leftarrow \text{MAC}_z(s_i)$
5. output $S_i = (s_i, z_i, t_i)$ to P_i

Rec($S_1, \dots, S_n$):

1. $z \leftarrow \text{RS.Rec}_1(z_1, \dots, z_n)$
2. set $i \in G$ if $t_i = \text{MAC}_z(s_i)$
3. $s \leftarrow \text{Shamir.Rec}_t(s_G)$

Privacy – Proof Intuition

Share(s):

1. sample MAC key $z \in X$
2. $(s_1, \ldots, s_n) \leftarrow \text{Shamir.Share}_t(s)$
3. $(z_1, \ldots, z_n) \leftarrow \text{Shamir.Share}_1(z)$
4. $t_i \leftarrow \text{MAC}_z(s_i)$
5. output $S_i = (s_i, z_i, t_i)$ to P_i

t-**privacy:** z uniform, independent of $s, s_1, \ldots, s_n$
$s_1, \ldots, s_t$ give no info on s, (privacy of Shamir.Share_t)
$t_1, \ldots, t_t$ functions only of $z, s_1, \ldots, s_t$
$\Rightarrow S_1, \ldots, S_t$ give no info on s

Robustness – Proof Intuition

$\mathrm{Rec}(S_1, \dots, S_n)$:

1. $z \leftarrow \mathrm{RS.Rec}_1(z_1, \dots, z_n)$
2. set $i \in G$ if $t_i = \mathrm{MAC}_z(s_i)$
3. $s \leftarrow \mathrm{Shamir.Rec}_t(s_G)$

(t, δ)-**robustness:** z correct, because $\mathrm{RS.Rec}_1$ decodes up to $(n-1)/2 = (2t+1-1)/2 = t$ errors

Adv_i **sees only** s_i, z_i, t_i
$\Rightarrow$ no info on z (privacy of $\mathrm{Shamir.Share}_1$)

MAC ε-secure
$\Rightarrow \Pr[i \in G \mid \widetilde{s_i} \neq s_i] \leq \varepsilon$
$\Rightarrow \Pr[G \subseteq H \cup P] \geq 1 - t \cdot \varepsilon$
$\Rightarrow \delta \leq t \cdot \varepsilon$

Possible MAC and Overhead Analysis

Remember: $\delta \leq t \cdot \varepsilon$

Assume: $m = |s|, \qquad 2 \cdot c = |z|, \qquad c = |t_i|, \qquad m = 2 \cdot d \cdot c$

$$\begin{array}{rccccl} \mathrm{MAC}: & (\mathbb{F}_{2^c})^2 & \times & \mathbb{F}_{2^m} & \to & \mathbb{F}_{2^c} \\ & (a, b), & & (m_1, \ldots, m_d) & \mapsto & \sum_{l=1}^{d} a^l \cdot m_l + b. \end{array}$$

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Set $c = k + \log(t \cdot m) = \widetilde{O}(k) \Rightarrow \delta \leq t \cdot m \cdot 2^{-k-\log(t \cdot m)-1} \cdot c^{-1} \leq 2^{-k}$

Overhead: $|z| + |t_i| = 2c + c = 3c = \widetilde{O}(k)$

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Optimality of Construction

Want to show:

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What we do: prove a stronger result!

$$\text{Scheme } (t, 2^{-k})\text{-robust against } \textit{oblivious} \text{ advs} \Rightarrow |s_i| \geq m + k - 4$$

local adv: $\widetilde{s_i} = \mathrm{Adv}_i(s_i)$
oblivious adv: $\widetilde{s_i} = \mathrm{Adv}_i(\varnothing)$

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Proof structure:

1. define an oblivious attack
2. link success of attack with share size

The Attack

Let s_{t+1} be the shortest share.

Specifications:

- "decide" whether to corrupt $P_1, \ldots, P_t$ (L) or $P_{t+2}, \ldots, P_n$ (R)
- sample secret $\widetilde{s} \leftarrow \mathcal{M}$, randomness $\widetilde{r} \leftarrow \mathcal{R}$
- run $(\widetilde{s_1}, \ldots, \widetilde{s_n}) \leftarrow \mathsf{Share}(\widetilde{s}, \widetilde{r})$
- if L, submit $\widetilde{s_1}, \ldots, \widetilde{s_t}$; if R, submit $\widetilde{s_{t+2}}, \ldots, \widetilde{s_n}$

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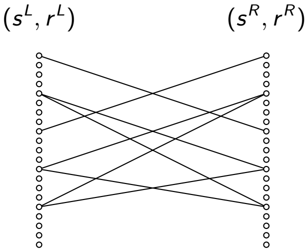
Intuition: hope that corrupt shares & s_{t+1} consistent with dishonest secret.

$$\mathrm{Rec}\left(\overbrace{s_1, \ \ldots, \ s_t, \ s_{t+1}}^{\text{partial sharing of } s^L}\!, \ \underbrace{\phantom{s_{t+1}}\!\!\!\!\!\!\!\!\!\!\!\! s_{t+2}, \ \ldots, \ s_n}_{\text{partial sharing of } s^R}\right) = ?$$

The Decision

Intuitively: find out whether L is more promising than R.

- Graph: (s^L, r^L) connected to (s^R, r^R) if:
  $\mathrm{Share}(s^L, r^L)_{t+1} = y = \mathrm{Share}(s^R, r^R)_{t+1}$, and $s^L \neq s^R$



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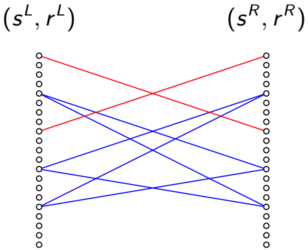
- Label edge with L (resp. R) if:
  $\mathsf{Rec}(s_1^L, \dots, s_t^L, y, s_{t+2}^R, \dots, s_n^R) \neq s^R$ resp. $\neq s^L$)



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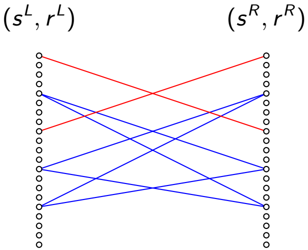
- Label edge with L (resp. R) if:
  $\text{Rec}(s_1^L, \ldots, s_t^L, y, s_{t+2}^R, \ldots, s_n^R) \neq s^R$ resp. $\neq s^L$)
- Decide L if #L-edges $\geq$ #R-edges.



The Success (WLOG assume L)

$$\overbrace{s_1,\ \ldots,\ s_t,\ s_{t+1}}^{s^L},\ s_{t+2},\ \ldots,\ s_n$$

$$\underbrace{s_1,\ \ldots,\ s_t}_{\widetilde{s}},\ \underbrace{s_{t+1},\ s_{t+2},\ \ldots,\ s_n}_{s^R}$$

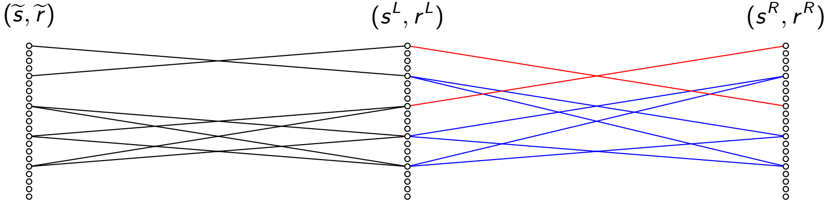
The Success (WLOG assume L)

$$\mathrm{Rec}\left(\overbrace{\underbrace{s_1,\ \ldots,\ s_t}_{\widetilde{s}},\ s_{t+1}}^{s^L},\ s_{t+2},\ \ldots,\ s_n\right) \neq s^R$$

with $\underbrace{s_{t+1},\ s_{t+2},\ \ldots,\ s_n}_{s^R}$

The Success (WLOG assume L)

$$\text{Rec}\left(\overbrace{\underbrace{s_1,\ \ldots,\ s_t}_{\widetilde{s}},\ s_{t+1}}^{s^L},\ \underbrace{\phantom{s_{t+1}},\ s_{t+2},\ \ldots,\ s_n}_{s^R}\right) \neq s^R$$



$\text{Share}(\widetilde{s}, \widetilde{r})_{\{1,\ldots,t\}} = \text{Share}(s^L, r^L)_{\{1,\ldots,t\}}$ $\quad$ $\text{Share}(s^L, r^L)_{t+1} = \text{Share}(s^R, r^R)_{t+1}$

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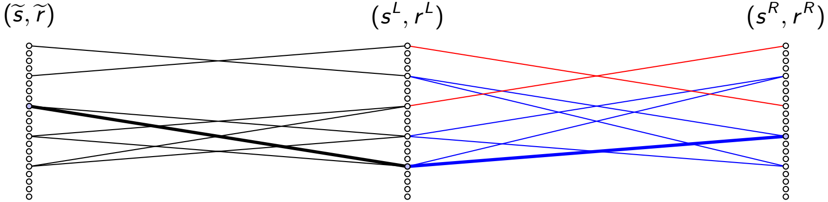

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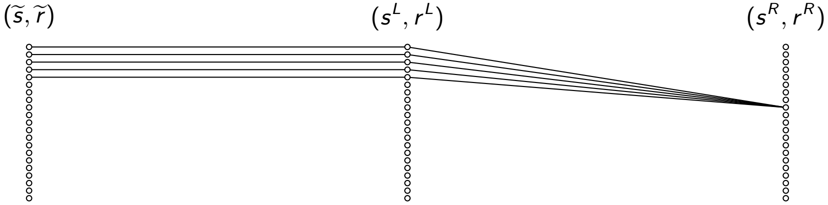
$$\delta = 2^{-k} \geq \Pr_{(\widetilde{s},\widetilde{r},s^R,r^R)}[\exists (s^L, r^L) \mid (\widetilde{s},\widetilde{r}) \text{—} (s^L, r^L) \overset{\mathsf{L}}{\text{—}} (s^R, r^R)]$$

Mass Distribution

For $a_1, \ldots, a_{t+1}$,
let $B_{a_1,\ldots,a_{t+1}} = \{(s^L, r^L) \mid \mathrm{Share}(s^L, r^L)_{\{1,\ldots,t+1\}} = a_1, \ldots, a_{t+1}\}$,
let $A_{a_1,\ldots,a_{t+1}} = \{(\widetilde{s}, \widetilde{r}) \mid \mathrm{Share}(\widetilde{s}, \widetilde{r})_{\{1,\ldots,t\}} = a_1, \ldots, a_t\}$.

Fact 1*: by reconstructability, $(s', r'), (s'', r'') \in B_{a_1,\ldots,a_{t+1}} \Rightarrow s' = s''$.



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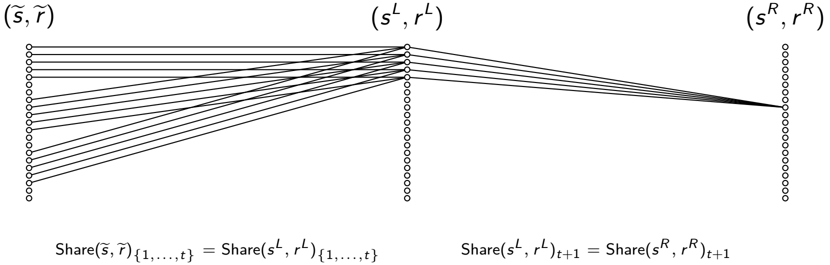
Fact 1*: by reconstructability, $(s', r'), (s'', r'') \in B_{a_1,\ldots,a_{t+1}} \Rightarrow s' = s''$.
Fact 2: by privacy, $|A_{a_1,\ldots,a_{t+1}}| \geq 2^m \cdot |B_{a_1,\ldots,a_{t+1}}|$.



Putting Things Together – Intuition

Actual analysis needs more correcting factors (loss of ~ 4 bits).

$$2^{-k} \geq \Pr_{(\widetilde{s},\widetilde{r},s^R,r^R)}[\exists (s^L, r^L) \mid (\widetilde{s},\widetilde{r}) \text{—} (s^L, r^L) \overset{\mathsf{L}}{\text{—}} (s^R, r^R)]$$

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$$\geq 2^m \cdot \Pr_{(s^L,r^L,s^R,r^R)}[(s^L, r^L) \overset{\mathsf{L}}{\text{---}} (s^R, r^R)]$$

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&\geq 2^{m-1} \cdot \Pr_{(s^L,r^L,s^R,r^R)}[(s^L,r^L)\text{—}(s^R,r^R)]\\
&\geq 2^{m-1} \cdot \sum_{a_{t+1}} \Pr_{(s^L,r^L,s^R,r^R)}[\mathrm{Share}(s^L,r^L)=a_{t+1},\mathrm{Share}(s^R,r^R)=a_{t+1}]\\
&\geq 2^{m-1} \cdot \sum_{a_{t+1}} \Pr_{(s,r)}[\mathrm{Share}(s,r)=a_{t+1}]^2
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&\geq 2^{m-1} \cdot \sum_{a_{t+1}} \Pr_{(s^L,r^L,s^R,r^R)}[\mathrm{Share}(s^L,r^L) = a_{t+1}, \mathrm{Share}(s^R,r^R) = a_{t+1}] \\
&\geq 2^{m-1} \cdot \sum_{a_{t+1}} \Pr_{(s,r)}[\mathrm{Share}(s,r) = a_{t+1}]^2 && \text{(Cauchy-Schwarz)} \\
&\geq 2^{m-1} \cdot 2^{-|s_{t+1}|} \left(\sum_{a_{t+1}} \Pr_{(s,r)}[\mathrm{Share}(s,r) = a_{t+1}] \cdot 1 \right)^2 \\
&= 2^{m-1} \cdot 2^{-|s_{t+1}|}
\end{aligned}
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2^{-k} &\geq \Pr_{(\widetilde{s},\widetilde{r},s^R,r^R)}[\exists(s^L,r^L) \mid (\widetilde{s},\widetilde{r})\text{---}(s^L,r^L)\overset{\mathsf{L}}{\text{---}}(s^R,r^R)] && \text{(Fact 1\&2)}\\
&\geq 2^m \cdot \Pr_{(s^L,r^L,s^R,r^R)}[(s^L,r^L)\overset{\mathsf{L}}{\text{---}}(s^R,r^R)]\\
&\geq 2^{m-1} \cdot \Pr_{(s^L,r^L,s^R,r^R)}[(s^L,r^L)\text{---}(s^R,r^R)]\\
&\geq 2^{m-1} \cdot \sum_{a_{t+1}} \Pr_{(s^L,r^L,s^R,r^R)}[\mathsf{Share}(s^L,r^L)=a_{t+1},\mathsf{Share}(s^R,r^R)=a_{t+1}]\\
&\geq 2^{m-1} \cdot \sum_{a_{t+1}} \Pr_{(s,r)}[\mathsf{Share}(s,r)=a_{t+1}]^2 && \text{(Cauchy-Schwarz)}\\
&\geq 2^{m-1} \cdot 2^{-|s_{t+1}|}\left(\sum_{a_{t+1}} \Pr_{(s,r)}[\mathsf{Share}(s,r)=a_{t+1}]\cdot 1\right)^2\\
&= 2^{m-1} \cdot 2^{-|s_{t+1}|}
\end{aligned}
$$

$$|s_{t+1}| \geq m + k - 1$$

Putting Things Together – Intuition

Actual analysis needs more correcting factors (loss of ~ 4 bits).

$$
\begin{aligned}
2^{-k} &\geq \Pr_{(\widetilde{s},\widetilde{r},s^R,r^R)}[\exists (s^L, r^L) \mid (\widetilde{s},\widetilde{r}) — (s^L, r^L) \overset{\mathsf{L}}{—} (s^R, r^R)] && \text{(Fact 1\&2)} \\
&\geq 2^m \cdot \Pr_{(s^L,r^L,s^R,r^R)}[(s^L, r^L) \overset{\mathsf{L}}{—} (s^R, r^R)] \\
&\geq 2^{m-1} \cdot \Pr_{(s^L,r^L,s^R,r^R)}[(s^L, r^L) — (s^R, r^R)] \\
&\geq 2^{m-1} \cdot \sum_{a_{t+1}} \Pr_{(s^L,r^L,s^R,r^R)}[\mathrm{Share}(s^L, r^L) = a_{t+1}, \mathrm{Share}(s^R, r^R) = a_{t+1}] \\
&\geq 2^{m-1} \cdot \sum_{a_{t+1}} \Pr_{(s,r)}[\mathrm{Share}(s, r) = a_{t+1}]^2 && \text{(Cauchy-Schwarz)} \\
&\geq 2^{m-1} \cdot 2^{-|s_{t+1}|} \left(\sum_{a_{t+1}} \Pr_{(s,r)}[\mathrm{Share}(s, r) = a_{t+1}] \cdot 1 \right)^2 \\
&= 2^{m-1} \cdot 2^{-|s_{t+1}|}
\end{aligned}
$$

$$|s_{t+1}| \geq m + k - 1 \quad ☺$$

Conclusion

Robust SS with $n = 2 \cdot t + 1$ players, eff. reconstruction. Share size:

model	construction	lower bound
standard	$m + \widetilde{O}(n + k)$	$m + k$
NEW: local adv.	$m + \widetilde{O}(k)$	$m + k - 4$

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Future:

- Locality in more complicated settings:
  - info theoretic MPC: circumvent lower bounds?
  - general MPC: more eff/practical protocols?
- standard RSSS: lower bound & construction matching?

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THANKS!

https://eprint.iacr.org/2014/909

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