On the Structure of Weakly Acyclic Games

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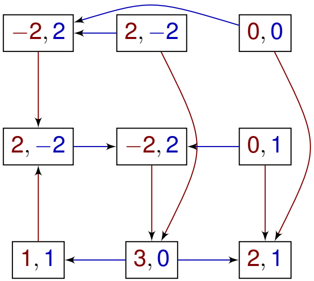
Best- and better-response dynamics

- Consider the best-response (or better-response) dynamics of a normal-form game:



- Pure NE = no outbound edges

Convergence to pure Nash

- *Existence* of pure Nash is good, but will the game converge to one?
- *Some* kinds of games always converge to pure Nash:
  - congestion/potential games [Rosenthal'73; Monderer&Shapley '96]
  - ordinal potential games (fully general for better-response)
  - dominance-solvable games [Moulin '79]
- But what of games that don't always converge?

Convergence to pure Nash

- Is this divergence interesting?

Play

	H	T
H	-1,1,0	1,-1,0
T	1,-1,0	-1,1,0

Stop

	H	T
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- *Weakly acyclic games*: every state *has a better-response path* to a pure Nash (no non-singleton sinks) [Young'93; Milchtaich'96]

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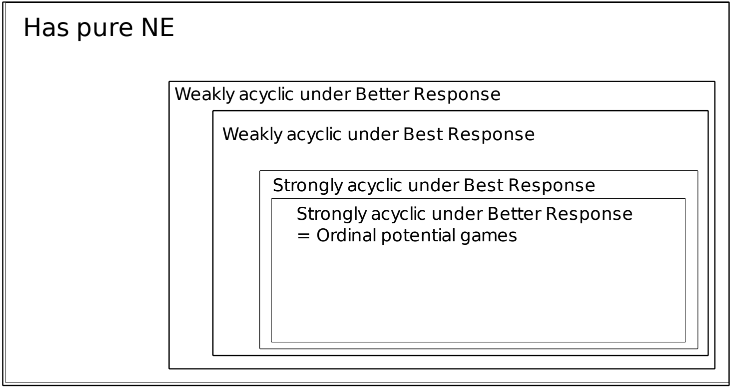
- Random player ordering $\Rightarrow$ *stochastic convergence a.s.*
- Other natural dynamics, like no-regret, also converge (Young, et al.)

The convergence map



Characterizing weak acyclicity

- Our contribution: combinatorial sufficient conditions that link subgame equilibria and weak acyclicity
  - Subgame: each player gets a subset $S_i' \subseteq S_i$ of her strategies

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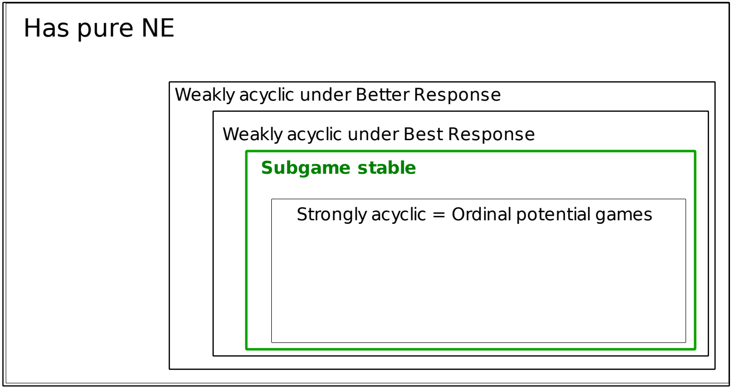
- Our contribution: combinatorial sufficient conditions that link subgame equilibria and weak acyclicity
  - Subgame: each player gets a subset $S'_i \subseteq S_i$ of her strategies
- Start with *subgame stability*: each subgame *has* pure NE
  - **Not** rare: *necessary* and *not* sufficient for ordinal potential
  - Originally from networking (BGP routing): subgame stability $\Leftrightarrow$ stability under failures

The general result

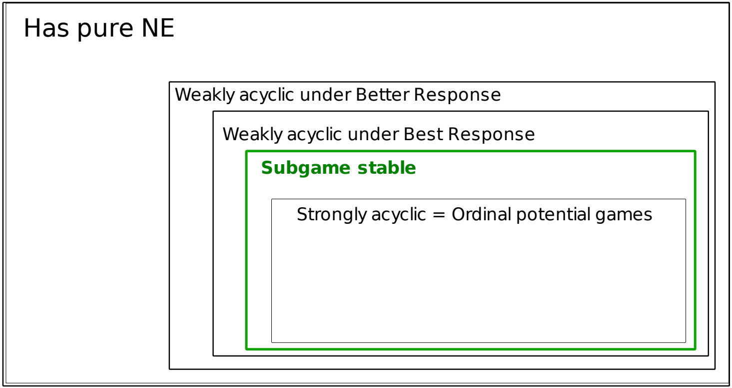
- 2-player game [Yamamori&Takahashi'02][1]:



[1] **Aaronson Conjecture**: w.h.p., the cute combinatorial lemma in your TCS paper was already proven in the '60s and published in Hungarian.

The general result

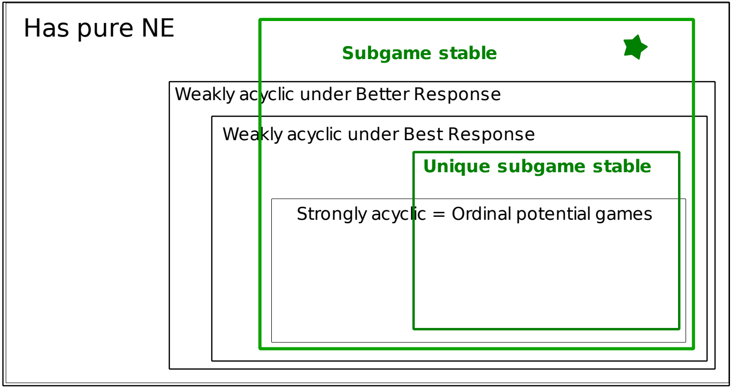
- 2-player game [Yamamori&Takahashi'02][2]:



[2] **Our corollary**: w.h.p., the cute combinatorial lemma in your AGT paper was already proven by economists, and published in Economese.

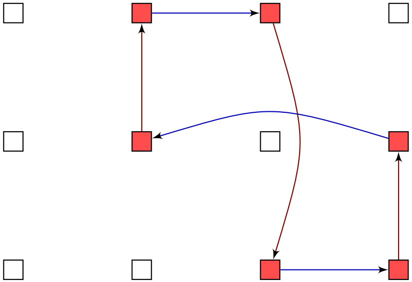
The general result

- *n*-player game:



- Unique Subgame Stability: each subgame has a *unique* pure NE

Two-player Subgame Stability [YT'02]:



- Not weakly acyclic $\Rightarrow$ BR dynamics has a *sink equilibrium* [Goemans, et al.'05] of size > 1

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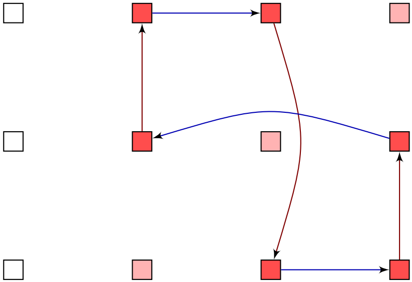

- Take the *span* of this component – subgame that includes all strategies used in the sink

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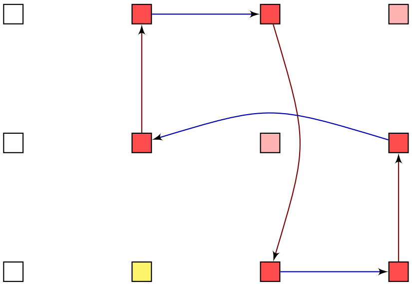

- This subgame has a pure NE, and the sink has a node in the same column
- The pure NE cannot be in the sink

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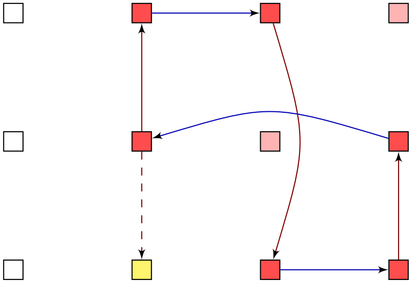

- But where does the BR by row player go?

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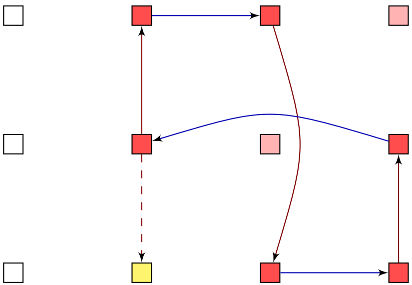

- Thus, 2-player SS
  $\Rightarrow$ Weak Acyclicity under Best Response
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n players: not so easy

- For 2 players, there is a sink state within 1 player's move from a pure NE
- For *n* players, within $\leq n - 1$ players' moves

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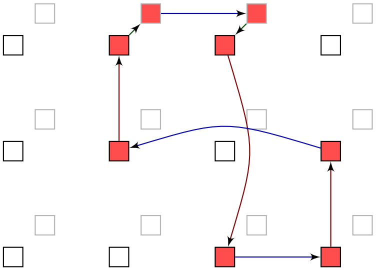
- For n players, within $\leq n-1$ players' moves
- Idea: fix players' strategies, one player at a time
- We'll need *unique* subgame stability

Unique SS ⇒ weak acyclicity (sketch)



- Similar: take the span of a **hypothetical big sink**, find its pure NE, follow best response to have one player match his strategy in the NE

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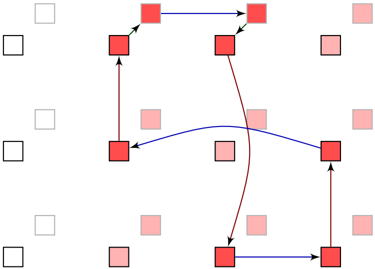

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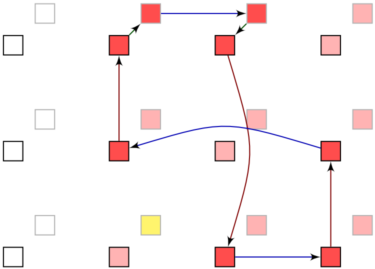

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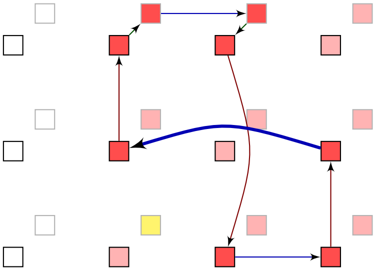

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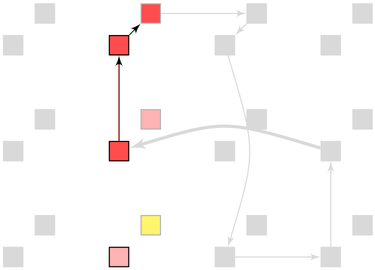

- Remove all of that player's strategies $\rightarrow$ smaller subgame, also has a pure Nash

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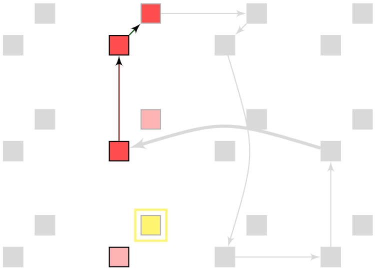

- Unique SS ⇒ it's the same Nash as before (non-trivial, but easy). Recurse to get closer.

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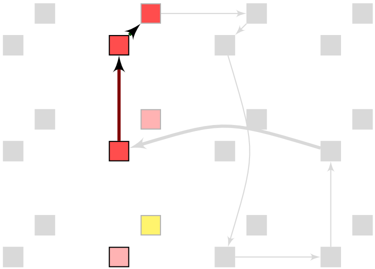

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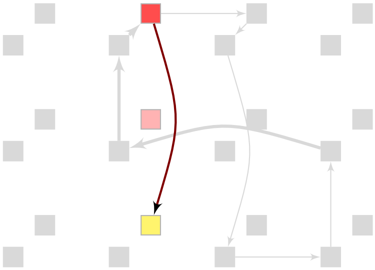

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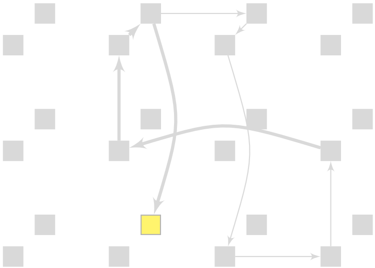

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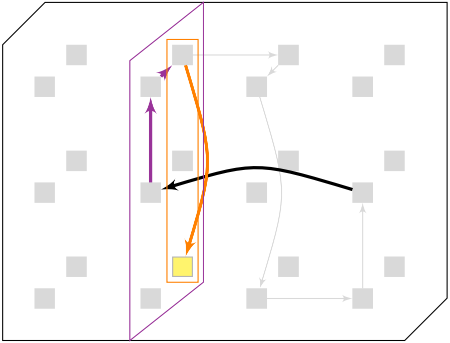
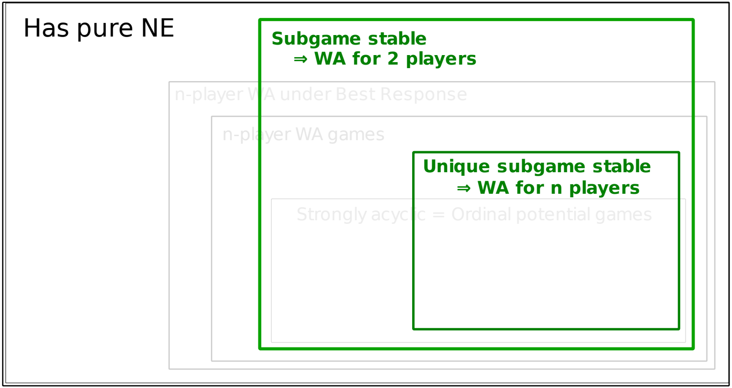


- Recursion builds up a path built of chunks of **BR paths from different subgames**: cheating? (no; see paper)

A finer subgame stability picture



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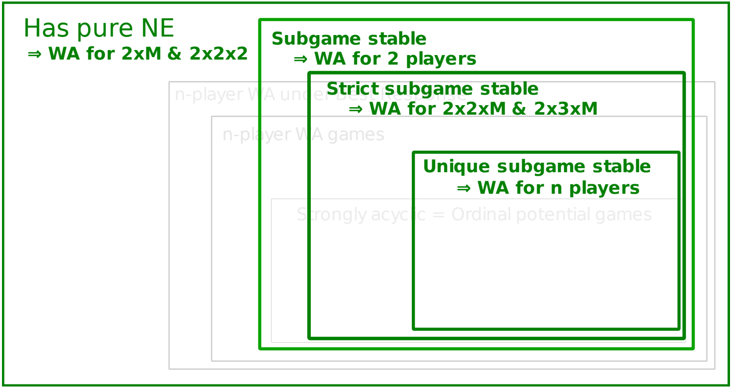

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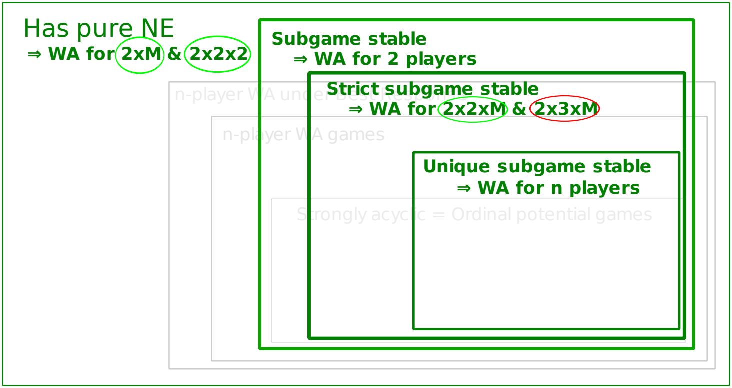

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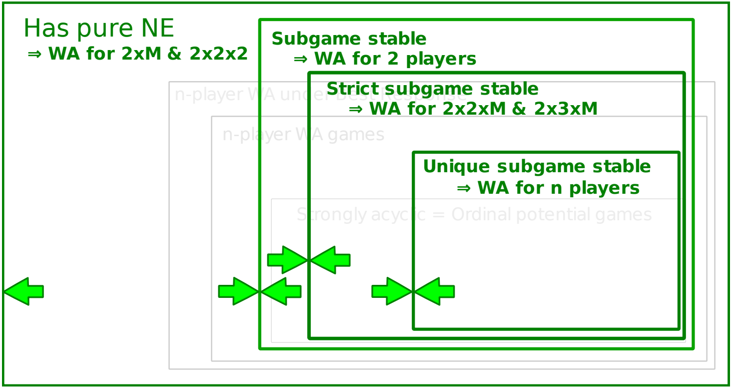

- The distinctions are tight w.r.t. game size

Open: Is there more structure to this space?

- Maybe there's an interesting intermediate property between SSS and USS?
  - (Our proof doesn't *quite* use full USS...)
- HasPNE $\supseteq$ SS $\supseteq$ SSS $\supseteq$ USS ...more to this hierarchy?

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- First combinatorial sufficient condition for weak acyclicity in general games
- Significantly lower complexity class:
  - [Mirrokni&Skopalik'09]: Weak acyclicity in several interesting succinct games is PSPACE-Complete.
  - For reasonable succinct games, all our conditions are low-ish in PH ($\Sigma_2 P$ and $\Sigma_3 P$)

Open problems

- Are there interesting game classes which obey USS by design, or can be tractably checked for USS?
- More broadly-applicable sufficient conditions of weak acyclicity?

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- Are there interesting game classes which obey USS by design, or can be tractably checked for USS?
- More broadly-applicable sufficient conditions of weak acyclicity?
- Weak acyclicity doesn't have to be tied to myopic dynamics...

Open problem: the elephant in the room

- Weakly Acyclic games converge stochastically
- Bad *worst-case convergence time*, even in nice, strongly acyclic games. E.g., exponential in network congestion games [F,Papadimitriou,Talwar'04]

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- What about the expected time until convergence, assuming, e.g., u.a.r. player orderings?
- Random walk mixing time for particularly-shaped directed graphs — maybe need more basic tools?
- “Good” news: no worse than exponential, but when is it actually *good*?

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- "Good" news: no worse than exponential, but when is it actually *good*?
- Interesting: Without strictness, clean exponentially-bad examples [Ferraioli, over lunch]. But ties are fragile...

Thank you