Today

Will prove more problems are NP-complete:

3D Matching
Generalized 3DM
Exact Cover
Subset Sum
Interval Sched with Deadlines and Release Times (ISDR)
Generalized 3DM

Given three sets, $A$, $B$, $C$, $|A| = |B| = k$
and triples $T_1, \ldots, T_n$, each with one element of $A$, $B$, and $C$

Do there exist $k$ pairwise disjoint triplets?

Equivalent: disjoint triplets that cover all of $A$ and $B$. 
Gen-3DM is NP-Complete

Clearly in NP, because can check a proposed matching. To prove NP-hard, will show 3-SAT ≤_p Gen-3DM.

Given an collection of clauses C_1, ..., C_k, each with at most 3 terms, on variables x_1, ..., x_n

produce sets A, B, C, and triples S_1, ..., S_m

that have matching iff the clauses are all satisfiable
Gen-3DM NP-Complete – variable gadgets

If these are only triples containing inner elements, must cover by all odd or all even triples
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3DM NP-Complete – variable gadgets

For variable $x_i$ in $d$ clauses, create gadget with $2d$ inner elements:

- $a_{i,1}, a_{i,2}, \ldots, a_{i,d}$
- $b_{i,1}, b_{i,2}, \ldots, b_{i,d}$

and $2d$ outer elements

- $c_{i,1,0}, c_{i,2,0}, \ldots, c_{i,d,0}$
- $c_{i,1,1}, c_{i,2,1}, \ldots, c_{i,d,1}$

and triples as shown:

- $(a_{i,k}, b_{i,k}, c_{i,k,0})$
- $(a_{i,k+1}, b_{i,k}, c_{i,k,1})$
3DM NP-Complete – variable gadgets

Interpret covering inner elements by odd sets as false.

Interpret covering inner elements by even sets as true

Expose $c_{i,*,0}$

Expose $c_{i,*,1}$
3DM NP-Complete – clause gadget

Say clause $C_j$ has form $x_1 \lor \overline{x}_2 \lor x_n$
Create two elements for the clause: $a^j$ and $b^j$

and create triples with these and terms that satisfy clause: $(a^j, b^j, c_{1,j,1}), (a^j, b^j, c_{2,j,0}), (a^j, b^j, c_{n,j,1})$. 
3DM NP-Complete – clause gadget

Say clause $C_j$ has form $x_1 \lor \overline{x_2} \lor x_n$

If these are only triples with the clause elements, must cover by a variable’s external element that satisfies clause, and variable gadgets enforce consistency
3DM NP-Complete – clause gadget

Say clause Cj has form $x_1 \lor \overline{x}_2 \lor x_n$

Each clause gets own external element for each variable
Truth assignment -> choice of triples at variable gadgets. Satisfying -> can choose a triple for each clause gadget.

Disjoint, and cover all of A and B.

\[ x_1 = 1 \quad x_2 = 0 \quad x_n = 0 \]
Truth assignment \(\rightarrow\) choice of triples at variable gadgets. Satisfying \(\rightarrow\) can choose a triple for each clause gadget.

Disjoint, and cover all of \(A\) and \(B\).

\[
x_1 = 1 \\
x_2 = 0 \\
x_n = 0
\]
Cover all internals \((A,B)\) once
\[-\text{truth assignment (var gadgets)}\]
Cover all clause internal elements \(-\text{satisfies clause}\)

\[
x_1 = 1 \\
x_2 = 0 \\
x_n = 0
\]