Algorithms, Lecture 3 on NP:
Nondeterministic Polynomial Time
Problem X is poly time reducible to Y
\[ X \leq_p Y \]
if can solve X using poly computation and
a poly number of calls to an algorithm
solving Y.

“Up to poly factors, X is at least as easy as Y”
“Up to poly factors, Y is harder than X”
Defined NP: decision (yes/no) problems can check certificates for yes answers in ptime

Have poly time proof checkers:
A poly time algorithm A so that
  if \( X(s) = \text{yes}, \)
    there exists a \( w \) so that \( A(s,w) = \text{yes}, \) and
  if \( X(s) = \text{no}, \)
    for all \( w, \) \( A(s,w) = \text{no} \)

Most problems of form: does there exist \( w \) so that satisfies conditions implied by \( s? \)
Last class:

Defined NP-Complete:

1. $X \in \text{NP}$, and
2. $X$ is NP-hard

$X$ is NP-hard if:

1. for all $Y \in \text{NP}, Y \leq_p X$, or
2. CircuitSat $\leq_p X$, or
3. $Y \leq_p X$, for some NP-Hard $Y$, such as SAT, Independent Set, Vertex Cover
Last class:

Proved CircuitSAT is NP-Hard.

Proved CircuitSAT $\leq_p$ SAT.

In fact, proved CircuitSAT $\leq_p$ 3-SAT

Where 3-SAT is SAT,

but each clause has at most 3 terms.
Last class:

Proved CircuitSAT is NP-Hard.

Proved CircuitSAT $\leq_p$ SAT.

In fact, proved CircuitSAT $\leq_p$ 3-SAT

Where 3-SAT is SAT,
    but each clause has at most 3 terms.
    Naturally follows from the construction.
SAT $\leq_p$ 3-SAT, a self-contained proof

A or B is equivalent to $\exists \ y$ s.t. (A or y) and (B or $\overline{y}$)

So, given a clause with more than 3 terms, like

$$x_1 \lor \overline{x_2} \lor x_3 \lor x_4$$

Introduce a new variable $y$, and replace with clauses:

$$x_1 \lor \overline{x_2} \lor y \quad \text{and} \quad \overline{y} \lor x_3 \lor x_4$$
SAT \leq_p 3-SAT, a self-contained proof

A or B is equivalent to \( \exists y \) s.t. \((A \lor y) \land (B \lor \neg y)\)

So, given a clause with more than 3 terms, like

\[ x_1 \lor \overline{x_2} \lor x_3 \lor x_4 \]

Introduce a new variable \( y \), and replace with clauses:

\[ x_1 \lor \overline{x_2} \lor y \text{ and } \overline{y} \lor x_3 \lor x_4 \]

2-SAT is in \( P \)
Also possible to force each variable to appear at most 3 times

Say a variable $x$ appears in $k$ clauses.

Replace $x$ with $k$ new variables, $x_1, \ldots, x_k$, one for each clause in which it appears.

Add clauses

$$x_1 \lor \overline{x_2}, \ x_2 \lor \overline{x_3}, \ldots, \ x_{k-1} \lor \overline{x_k}, \ x_k \lor \overline{x_1}$$

Only satisfied if all are equal.
Today

Will prove more problems are NP-complete:

3-coloring
Hamiltonian Cycle
Travelling Salesperson Problem
k-Coloring

Given a graph $G = (V,E)$, does there exist $f : V \rightarrow \{1,2,..., k\}$ (colors)

So that for all $(u,v) \in E$ $f(u) \neq f(v)$ ?

3-colorable

Not 3-colorable
k-Coloring

Given a graph $G = (V,E)$, does there exist $f : V \rightarrow \{1,2,\ldots,k\}$ (colors)
So that for all $(u,v) \in E$ $f(u) \neq f(v)$ ?

3-colorable

Not 3-colorable
k-Coloring is NP-Complete

Clearly in NP, because can check a proposed coloring
To prove NP-hard, will show $3\text{-SAT} \leq_p 3\text{-Coloring}$

Given a collection of clauses $C_1, \ldots, C_k$, each with at most 3 terms, on variables $x_1, \ldots, x_n$

produce graph $G = (V,E)$ that is
3-colorable iff the clauses are satisfiable
3-Coloring is NP-Complete – variable gadgets

Create 3 special nodes: T, F, B (base), and one node for each term: \( x_i \) and \( \overline{x_i} \)

In every 3-coloring, one of \( x_i \) and \( \overline{x_i} \) is colored T and one is colored F
3-Coloring is NP-Complete – variable gadgets

Create 3 special nodes: T, F, B (base), and one node for each term: $x_i$ and $\overline{x_i}$

In every 3-coloring, one of $x_i$ and $\overline{x_i}$ is colored T and one is colored F
3-Coloring is NP-Complete – clause gadgets

Consider clause $x_1 \lor \overline{x_2} \lor x_n$

Claim: 3-colorable iff terms colored to satisfy clause
3-Coloring is NP-Complete – clause gadgets

Claim: 3-colorable iff terms colored to satisfy clause

1. If terms all colored F, then cannot 3-color
3-Coloring is NP-Complete – clause gadgets

Claim: 3-colorable iff terms colored to satisfy clause

1. If terms all colored F, then cannot 3-color
3-Coloring is NP-Complete – clause gadgets

Claim: 3-colorable iff terms colored to satisfy clause

1. If terms all colored F, then cannot 3-color
3-Coloring is NP-Complete – clause gadgets

Claim: 3-colorable iff terms colored to satisfy clause

2. If some term true, can 3-color
3-Coloring is NP-Complete – clause gadgets

Claim: 3-colorable iff terms colored to satisfy clause

2. If some term true, can 3-color
3-Coloring is NP-Complete – clause gadgets

Claim: 3-colorable iff terms colored to satisfy clause

2. If some term true, can 3-color
3-Coloring is NP-Complete

3-colorable iff satisfiable

\[ x_1 \lor \overline{x_2} \lor x_n \]
3-Coloring is NP-Complete

$T$ $F$

3-colorable iff satisfiable

$x_1 \lor \overline{x}_2 \lor x_n$

$\overline{x}_2 \lor \overline{x}_3 \lor x_n$
Hamiltonian Cycle:
A cycle in a graph that hits each vertex once.

Directed Hamiltonian Cycle:
same, but in a directed graph
Directed Ham Cycle is NP-Complete

Clearly in NP, because can check if a cycle is Hamiltonian

To prove NP-hard, will show

\[ \text{SAT} \leq_p \text{Directed Ham Cycle} \]

Produce directed graph \( G = (V,E) \) that has Ham Cycle iff the clauses are satisfiable
Start: create graph with $2^n$ Ham Cycles, then create gadgets to restrict them.
Start: create graph with $2^n$ Ham Cycles, then create gadgets to restrict them.
Start: create graph with $2^n$ Ham Cycles, then create gadgets to restrict them

Must go top-to-bottom, and can traverse each row left-to-right (True) or right-to-left (False)
Start: create graph with $2^n$ Ham Cycles, then create gadgets to restrict them

Must go top-to-bottom, and can traverse each row left-to-right (True) or right-to-left (False)
Clause gadgets

clause \( x_1 \) forces traverse first row

Require: no other edges touch vertices in gadget
Clause gadgets

Forces traverse 1 ➡️, or 2⬅️, or n ➡️
Clause gadgets

clause

\[ x_1 \lor \overline{x}_2 \lor x_n \]

Forces traverse 1, or 2, or n

Require: no other edges touch vertices in gadget
Clause gadgets

clause

$x_1 \lor \overline{x_2} \lor x_n$

To see must come back to same row, note that if do not is no Hamiltonian path through unused down-link
Clause gadgets

clause

$\text{Forces traverse } 1 \rightarrow, \text{ or } 2 \leftarrow, \text{ or } n \rightarrow$
Ham cycle iff satisfiable

Pf. If satisfiable, traverse in order indicated by vars, picking up each clause once using some true term.
Ham cycle iff satisfiable

Pf. If Ham Cycle, must go top to bot

assign vars by direction

if visit each clause node, then is made true by term on row from which make the visit.
Directed Ham Cycle $\leq_p$ Ham Cycle

1. In directed problem, answer same if reverse all arrows.

2. To transform to undirected, replace each vertex $v$ with three vertices: $V_{\text{in}}, V_{\text{base}}, V_{\text{out}}$

Replace directed $(u,v)$ edge with $(u_{\text{out}}, v_{\text{in}})$
Claim: If these are only edges to $u_b$, then in every Hamiltonian cycle $u_b$ must be adjacent to $u_{in}$

Proof: if it is not, then once enter $u_b$ can not get out
Directed Ham Cycle $\leq_p$ Ham Cycle

Replace directed \((u,v)\) edge with \((u_{\text{out}}, v_{\text{in}})\)

Directed Ham Cycle in original $\rightarrow$ Ham Cycle
Directed Ham Cycle $\leq_p$ Ham Cycle

Replace directed $(u,v)$ edge with $(u_{\text{out}}, v_{\text{in}})$

**Lemma:**

Every Ham Cycle in the undirected graph must go in, base, out, in, base, out, in, base, out, etc, must correspond to a Ham Cyc in directed graph
TSP (Travelling Salesperson Problem)

Given $n$ locations, a distance function $d(u,v)$ and a total distance $D$, does there exist a tour through all locations of total distance at most $L$?

http://www.tsp.gatech.edu/usa13509/usa13509_info.html
TSP (Travelling Salesperson Problem)

Given n locations, a distance function \(d(u,v)\) and a total distance D, does there exist a tour through all locations of total distance at most L?

http://www.tsp.gatech.edu/usa13509/usa13509_sol.html
An optimal solution for **RL5915** is given by the following tour, which has length 565530.

http://www.tsp.gatech.edu/rl5915/rl5915_sol.html
TSP is NP-complete

Ham Cycle $\leq_p$ TSP

Given graph $G = (V,E)$,
create one location for each vertex,

$$d(u,v) = 1 \text{ if } (u,v) \in E$$
2 otherwise

Target distance $= |V|$

A tour of all locations that returns to start and has total length $|V|$ must use exactly $|V|$ edges of $G$.
TSP is NP-complete

Ham Cycle \leq_p TSP

Given graph $G = (V,E)$,
create one location for each vertex,

$$d(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 2 & \text{otherwise} \end{cases}$$

This is an abstract distance function.
Satisfies $d(u,v) = d(v,u)$ for all $u,v$ and
$$d(u,w) \leq d(u,v) + d(v,w)$$ for all $u, v, w$
TSP is NP-complete

Ham Cycle $\leq_p$ TSP

Given graph $G = (V,E)$,
create one location for each vertex,

$$d(u,v) = 1 \text{ if } (u,v) \in E$$
$$2 \text{ otherwise}$$

This is an abstract distance function.

Remains NP-hard for integer points in plane.
Issue with Planar TSP

If input is locations of points, instead of distances

The problem is not known to be in NP, because do not know if can compare distances in polynomial time.

For integers $x_1, ..., x_n$ integer $t$, do not have poly time algorithm to test if

$$\sum_i \sqrt{x_i} \leq t$$
Classic Nintendo Games are (NP-)Hard

Greg Aloupis*  Erik D. Demaine†  Alan Guo‡‡

March 9, 2012

Abstract

We prove NP-hardness results for five of Nintendo’s largest video game franchises: Mario, Donkey Kong, Legend of Zelda, Metroid, and Pokémon. Our results apply to Super Mario Bros. 1, 3, Lost Levels, and Super Mario World; Donkey Kong Country 1–3; all Legend of Zelda games except Zelda II: The Adventure of Link; all Metroid games; and all Pokémon role-playing games. For Mario and Donkey Kong, we show NP-completeness. In addition, we observe that several games in the Zelda series are PSPACE-complete.

Figure 4: Variable gadget for Mario

Figure 5: Clause gadget for Mario
Candy Crush is NP-hard

Toby Walsh

NICTA and University of NSW, Sydney, Australia

Abstract

We prove that playing Candy Crush to achieve a given score in a fixed number of swaps is NP-hard.

Keywords: computational complexity, NP-completeness, Candy Crush.
Bejeweled, Candy Crush and other Match-Three Games are (NP-)Hard

L. Gualà¹, S. Leucci², and E. Natale³

¹Università degli Studi di Roma Tor Vergata
guala@mat.uniroma2.it
²Università degli Studi dell'Aquila
stefano.leucci@univaq.it
³Sapienza Università di Roma
natale@di.uniroma1.it

March 25, 2014