Lecture 4 on NP
Today

Will prove more problems are NP-complete:

3D Matching
Generalized 3DM
Exact Cover
Subset Sum
Interval Sched with Deadlines and Release Times (ISDR)
Generalized 3DM

Given three sets, $A$, $B$, $C$, $|A| = |B| = k$
and triples $T_1, \ldots, T_n$, each with one element of $A$, $B$, and $C$

Do there exist $k$ pairwise disjoint triplets?

Equivalent: disjoint triplets that cover all of $A$ and $B$. 
Gen-3DM is NP-Complete

Clearly in NP, because can check a proposed matching. To prove NP-hard, will show $3\text{-SAT} \leq_p \text{Gen-3DM}$.

Given an collection of clauses $C_1, ..., C_k$, each with at most 3 terms, on variables $x_1, ..., x_n$

produce sets $A$, $B$, $C$, and triples $S_1, ..., S_m$

that have matching iff the clauses are all satisfiable
Gen-3DM NP-Complete – variable gadgets

If these are only triples containing inner elements, must cover by all odd or all even triples.
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3DM NP-Complete – variable gadgets

For variable $x_i$ in $d$ clauses, create gadget with 2d inner elements:

$a_{i,1}, a_{i,2}, ..., a_{i,d}$

$b_{i,1}, b_{i,2}, ..., b_{i,d}$

and 2d outer elements

$c_{i,1,0}, c_{i,2,0}, ..., c_{i,d,0}, c_{i,1,1}, c_{i,2,1}, ..., c_{i,d,1}$

and triples as shown:

$(a_{i,k}, b_{i,k}, c_{i,k,0}), (a_{i,k+1}, b_{i,k}, c_{i,k,1})$
3DM NP-Complete – variable gadgets

Interpret covering inner elements by odd sets as false.

Interpret covering inner elements by even sets as true

Expose $c_{i,*,0}$

Expose $c_{i,*,1}$
3DM NP-Complete – clause gadget

Say clause \(C^j\) has form \(x_1 \lor \overline{x_2} \lor x_n\)

Create two elements for the clause: \(a^j\) and \(b^j\)

and create triples with these and terms that satisfy clause:

\((a^j, b^j, c_{1,j,1}), (a^j, b^j, c_{2,j,0}), (a^j, b^j, c_{n,j,1})\),
3DM NP-Complete – clause gadget

Say clause C_j has form $x_1 \lor \overline{x}_2 \lor x_n$

If these are only triples with the clause elements, must cover by a variable’s external element that satisfies clause, and variable gadgets enforce consistency.
3DM NP-Complete – clause gadget

Say clause Cj has form $x_1 \lor \overline{x}_2 \lor x_n$

Each clause gets own external element for each variable
Truth assignment -> choice of triples at variable gadgets. Satisfying -> can choose a triple for each clause gadget.

Disjoint, and cover all of A and B.

\[ x_1 = 1 \quad x_2 = 0 \quad x_n = 0 \]
Truth assignment -> choice of triples at variable gadgets. Satisfying -> can choose a triple for each clause gadget.

Disjoint, and cover all of A and B.
Cover all internals (A,B) once
- truth assignment (var gadgets)
Cover all clause internal elements -> satisfies clause

\[ \begin{align*}
 x_1 &= 1 \\
 x_2 &= 0 \\
 x_n &= 0
\end{align*} \]