10/10 - Percolation I

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10:51 AM

Are given a graph, typically infinite, a probability p, and a distinguished node, "O"

Keep each edge with prob p.
Call kept edges "open" and others "closed"

In infinite case, ask for prob that O is in an infinite component

In finite case, ask for prob that O is connected to boundary, or that there is a large component containing a constant fraction ~~of~~ the vertices.

Will begin with infinite case, as is easier.

Two infinite graphs:

Lattice $\mathbb{Z}^2$: $V = \mathbb{Z} \times \mathbb{Z}$

$E = \{((a,b),(c,d)) : |a-c| + |b-d| = 1\}$

distinguish $(0,0)$

Infinite binary tree: T_2

$V = \{0,1\}^* \leftarrow$ finite sequences of 0's and 1's

$E = (x,y) \in U$ s.t. $\text{length}(y) = \text{length}(x)+1$

and $y = x0$ or $y = x1$

distinguish node ϕ ← empty set

Let C be component of distinguished node.

Define $\Theta(p) = \Pr_p[|C| = \infty]$

the prob that C is infinite

Critical threshold is p_c s.t.

$$p_c = \sup\{p \mid \Theta(p) = 0\}$$

Is a phase change threshold
Can show that

for $p < p_c$ $\Pr[\exists \text{ an inf cluster}] = 0$

$p > p_c$ $\Pr[\exists \text{ an inf cluster}] = 1$

By Kolmogorov's 0/1 Law

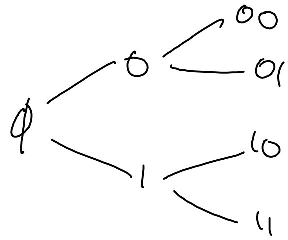
For complete binary tree

$\Pr[\phi$ is in inf component$]$

$$= \Pr\Big[(\text{edge } (\phi,0) \text{ appears} \wedge 0 \text{ in inf comp}) \vee$$

$$(\text{edge}(\phi,1) \text{ appears} \wedge 1 \text{ in inf comp})\Big]$$

By self-similarity, $\Pr[1 \text{ in inf comp}] = \Pr[\phi \text{ in inf comp}]$



So, by independence, and inclusion-exclusion

$$\theta(p) = 2p\theta(p) - (p\theta(p))^2$$

So, either $\theta(p) = 0$, or

$$1 = 2p - p^2\theta(p) \implies \theta(p) = \frac{2p-1}{p^2}$$

So, for $p \leq \frac{1}{2}$, is no solution

and, for $p > \frac{1}{2}$, $\theta(p) > 0$. $\implies p_c = \frac{1}{2}$

Let's treat as limit of finite trees

Let T_2^d be binary tree of depth d.

Consider $\theta^d(p)$ = prob is path from ϕ to a leaf in T_2^d

Consider $\lim_{d \to \infty} \theta^d(p)$ should $= \theta(p)$

By union bound,

$\theta^d(p) = \Pr[$ is path from ϕ to a node in $\{0,1\}^d]$

$$\leq \sum_{v \in \{0,1\}^d} \Pr[\text{is a path from } \emptyset \text{ to } v]$$

$$= \sum_{v \in \{0,1\}^d} p^d = 2^d p^d = (2p)^d \to 0$$

if $p < \frac{1}{2}$

So, $p_c \leq \frac{1}{2}$

To prove $p_c \geq \frac{1}{2}$,

Note $\theta^0(p) = 1$, as just one node

$$\theta^1(p) = 2p - p^2$$

$$\theta^d(p) = 2p\,\theta^{d-1}(p) - \left(p\,\theta^{d-1}(p)\right)^2$$

what is $\lim_{d \to \infty} \theta^d(p)$?

Claim: if $\theta^{d-1}(p) > \frac{2p-1}{p^2}$, then

$$\theta^d(p) > \frac{2p-1}{p^2}$$

So, approaches limit > 0

<u>pf of claim</u>

Consider $f(x) = 2px - (px)^2$

We know $f(x^*) = x^*$ for $x^* = \frac{2p-1}{p^2}$

and $f'(x) = 2p - 2p^2 x > 0$ for $0 \le x \le 1$
so, is monotone increasing.

$\Rightarrow$ for $x > x^*$ $f(x) > f(x^*)$

Regular Grid

is known that $p_c = \frac{1}{2}$. Not easy

Will prove $\frac{1}{3} \le p_c \le \frac{2}{3}$

<u>Thm</u> $\frac{1}{3} \le p_c$

<u>Def</u> $\sigma(n) = \#$ loop-free walks from $\bar{O}$ of length n., $S(n) =$ set of loop-free paths of length n

<u>Prop</u> $\sigma(n) \le 4 \cdot 3^{n-1}$

<u>pf of Thm</u>

If 0 is in inf component, then is a path from the origin of every length.

$\theta(p) \le \Pr$ [there is a path $P \in S(n)$ all of whose edges open]

$\le \sum \Pr$ [all edges in P open]

$p \in S(n)$

$$= \sum_{p \in S(n)} p^n = |S(n)|\, p^n \leq 4 \cdot 3^{n-1} p^n$$

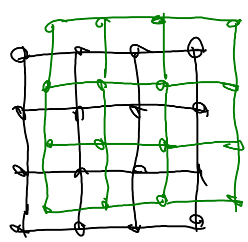
$$\to 0 \text{ if } p < \tfrac{1}{3}$$

So $\theta(p) = 0$ if $p < \frac{1}{3}$.

Thm $p < \frac{2}{3}$.

Will use dual graph:

vertex in center of every grid square,
dual edges cross primal edges



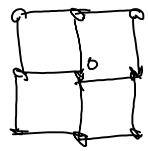
Set dual edge open $\iff$ consep primal edge closed.

<u>Prop</u>
0 is <u>not</u> in inf cluster
$\Updownarrow$
Dual contains open cycle around 0.

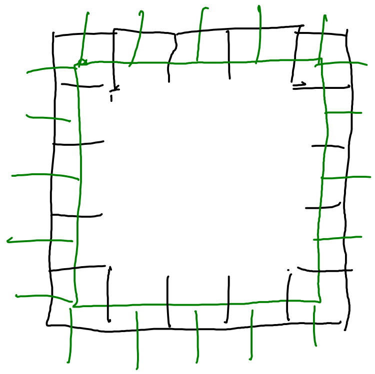
Let B_m = box ~~of~~ edges on $[-m,m] \times [-m,m]$

X_m = event all edges in B_m are open

Y_m = event Dual does not contain open cycle around B_m



X_m all these present



Y_m depends on these

$X_m \wedge Y_m \Rightarrow O$ in inf component

Let's show this probability is non-zero.

First, are independent, and $\Pr[X_m] > 0$

So, just need to show $\Pr[Y_m] > 0$

Will focus on $\Pr[\overline{Y_m}] < 1$

Let $C(n)$ = # cycles around origin of length n (simple).

How many? First, must pass through $(\frac{1}{2}, \frac{1}{2}+k)$

for some $0 \le k \le n$.

From there, is a self-avoiding walk of

length $n-1$ (up to last step)

So, $|C(n)| \le n \cdot S(n-1) \le 4n \cdot 3^{n-2}$

A cycle around B_m has length at least $8m$

So, $$\Pr[\bar{Y}_m] \le \sum_{n \ge 8m} \sum_{C \in C(n)} \Pr[\text{all dual edges of } C \text{ open}]$$

$$= \sum_{n \ge 8m} |C(n)| \cdot p^n \le \sum_{n \ge 8m} 4n \cdot 3^{n-1} p^n$$

$$= \sum_{n \ge 8m} \left(\frac{4}{3}\right) n (3p)^n \longrightarrow 0 \text{ for } m \text{ suff large}$$

$$\left(= \frac{4}{3} \left[\frac{(8m)(3p)^{8m}}{1-3p} + \frac{(3p)^{8m+1}}{(1-3p)^2} \right] \right)$$