10/12 Percolation, part II

Thursday, October 12, 2006
12:44 PM

Let L_2 denote graph on inf 2d grid.

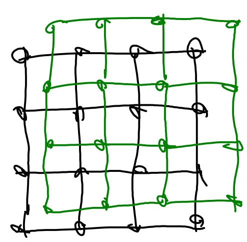
let G be a subgraph of L_2 chosen by including each edge with probability p.

$\theta(p) = \Pr[0 \text{ is in infinite component}]$

Today: 1. $p > \frac{2}{3} \Rightarrow \theta(p) > 0$
2. $p \le \frac{1}{2} \Rightarrow \theta(p) = 0$

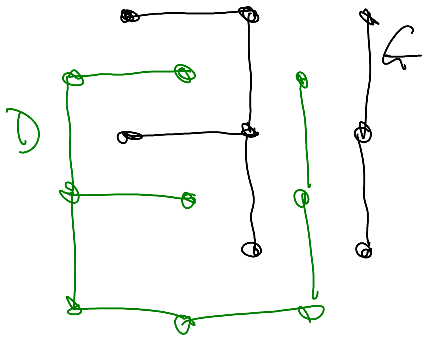
Key technique is dual graph

Dual to L_2 is also L_2, but shifted by $(\frac{1}{2}, \frac{1}{2})$
vertex in center of every grid square,
dual edges cross primal edges



let D be graph in L_2 containing duals of edges not in G

Ex.



Prop

0 is <u>not</u> in inf cluster in G

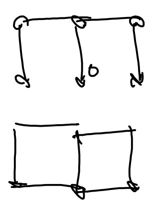
$\Updownarrow$

Dual contains open cycle around 0.

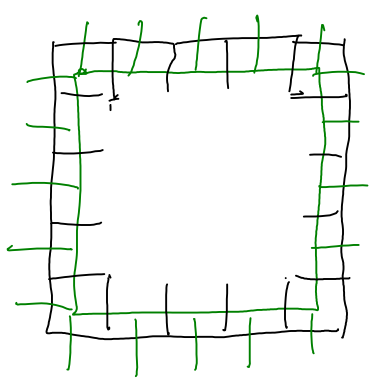
Let B_m = box ~~of~~ edges on $[-m,m]\times[-m,m]$

X_m = event all edges in B_m are open

Y_m = event Dual does not contain open cycle around B_m



X_m all these present



Y_m depends on these

$X_m \wedge Y_m \Rightarrow 0$ in inf component

Let's show this probability is non-zero.

First, are independent, and $\Pr[X_n] > 0$

So, just need to show $\Pr[Y_m] > 0$

Will focus on $\Pr[\overline{Y_m}] < 1$

Let $C(n)$ = # cycles around origin of length n (simple).

How many? First, must pass through $(\frac{1}{2}, \frac{1}{2}+k)$ for some $0 \le k \le n$.

From there, is a self-avoiding walk of length $n-1$ (up to last step)

So, $|C(n)| \le n \cdot S(n-1) \le 4n \cdot 3^{n-2}$

A cycle around B_m has length at least $8m$

So, $\Pr[\overline{Y_m}] \le \sum_{n \ge 8m} \sum_{C \in C(n)} \Pr[\text{all dual edges of } C \text{ open}]$

$$= \sum_{n \ge 8m} |C(n)| \cdot p^n \le \sum_{n \ge 8m} 4n \cdot 3^{n-1} p^n$$

$$= \sum_{n \ge 8m} \left(\frac{4}{3}\right) n (3p)^n \longrightarrow 0 \text{ for } m \text{ suff large}$$

$$\left(= \frac{4}{3}\left[\frac{(8m)(3p)^{8m}}{1-3p} + \frac{(3p)^{8m+1}}{(1-3p)^2} \right] \right)$$

Next: $\theta(\frac{1}{2}) = 0$

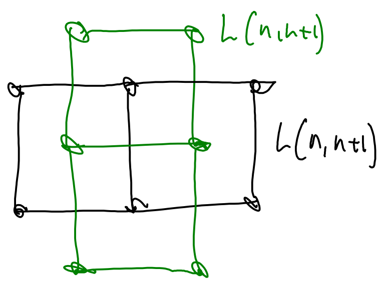
proof idea: show that D almost definitely contains a cycle around the origin

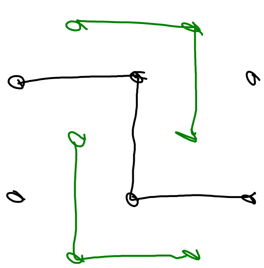
Step 1: let $L(n,m)$ be grid with n rows m cols

G sampled from $L(n,n+1)$

<u>Lem 1</u> Prob [G contains left-right path] $= \frac{1}{2}$

<u>Proof</u> consider the dual graph D in $L(n+1,n)$



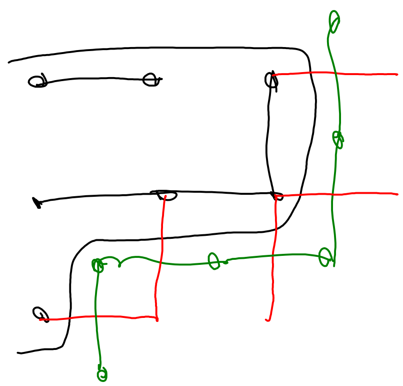


Either G contains a left-right path, or D contains a top-bottom path.

Clearly, if G contains a LR path, D does not contain a Top-Bot path.

To go other way

if G does not contain a LR path,
consider boundary of all nodes connected
to left side



none of the
boundary edges are in G,
so their duals are
in D

And, they connect the top to the bottom.

Want to build on paths like these

We know $\Pr[\, L(n,n)$ has open L-R path $] \geq \frac{1}{2}$

and $\Pr[\, L(n,n)$ has open T-B path $] \geq \frac{1}{2}$

What about

$\Pr[\, L(n,n)$ has open L-R path and open T-B path $] \geq ?$

Would be easy if independent,
but are not.

Still, seem positively correlated.

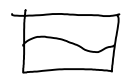
Say an event is increasing
if for all graphs in which true, if
make more edges open will remain true

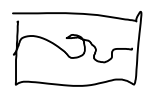
Ex: contains LR path.

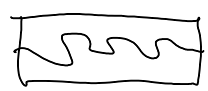
Thm FKG inequality.
If X and Y are increasing events

$$\Pr[X \wedge Y] \geq \Pr[X]\Pr[Y]$$

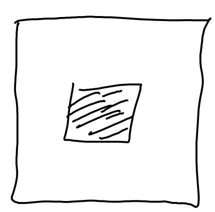
Def:

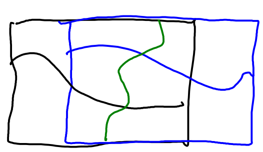
$\tau_1 = \Pr[L(n, \frac{3n}{2})$ has open LR path$]$ 

$\tau_2 = \Pr[L(n, 2n)$ has open LR path$]$ 

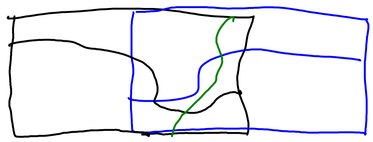
$\tau_3 = \Pr[L(n, 3n)$ has open LR path$]$ 

$\tau_4 = \Pr[B(3n) - B(n)$ contains open cycle around $0]$



1. $\tau_2 \geq \frac{1}{2}\tau_1^2$ 

2. $\tau_3 \geq \frac{1}{2}\tau_2^2$

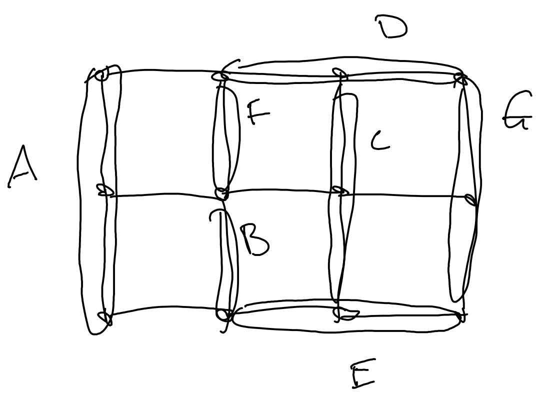


3. $\tau_4 \geq \tau_3^4$ So, $\tau_4 \geq \frac{1}{8}\tau_1^{16}$

What is τ_1 ?

Here is a heuristic argument that $\tau_1 > 0$.
It's not quite formal, but is close to the formal argument.

Divide the $n \times \frac{3}{2}n$ grid into regions as follows:



We will show that with some non-zero probability, there is a DE path that crosses an AC path and a FG path.

1. Define an ABC path to be an AC path s.t. the last time it crosses BF (before C), it crosses at B.

By symmetry $\Pr[ABC] \geq \frac{1}{2}\Pr[AC] \geq \frac{1}{4}$

2. Define an ABC-good DE path to be a DE path that crosses an ABC path

as each DE path, or its reflection, crosses the BC portion of a BC path, it should be (not formal)

$Pr[\exists \text{ ABC path and an ABC-good DE path}]$

$\geq \frac{1}{2} Pr[\exists \text{ ABC path}] \cdot Pr[\exists \text{ DE path}]$

$\geq \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

And, $Pr[\text{FG path}] \geq \frac{1}{4}$

So, $\tau_1 \geq \frac{1}{32}$

$\tau_4 \geq \frac{1}{8} \tau_1^{16} = \frac{1}{2^3} \left(\frac{1}{2^5}\right)^{16} = \frac{1}{2^{83}}$