10/17: Erdos-Renyi Graphs

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Random graphs.
Specify n = # vertices
p = prob of edge

For each $i < j$, include edge (i, j) with prob p, independently for each edge.

Let $G(n, p)$ denote this distribution.

We will see that the following phenomena hold with high probability

p	property
$< \frac{1}{n}$	disconnected, small tree-like components
$> \frac{1}{n}$	$\exists$ connected component on const frac of vertices
$< \frac{\ln(n)}{n}$	isolated vertices
$> \frac{\ln(n)}{n}$	connected
$< \frac{n^{\frac{1}{k}}}{n}$	neighborhoods of most vertices trees to depth $k/2$

First, expect $\binom{n}{2} p$ edges,

so for $p = \frac{1}{n}$, expect $\frac{n}{2}$ edges --

[illegible]

so expected degree of each vertex is 1

To bound chance it differs,
apply Chernoff - Hoeffding bounds:

Let $X_1, \dots, X_k$ be ind rand variables s.t.

$$\Pr[X_i = 1] = p_i$$

$$\Pr[X_i = 0] = 1 - p_i.$$

Let $X = \sum_i X_i$

Let $\mu = \sum_i p_i = E[X]$

Then $\forall\, \delta < 1$

$$\Pr[X < (1-\delta)\mu] < e^{-\frac{\delta^2 \mu}{2}}$$

$$\Pr[X > (1+\delta)\mu] < e^{-\frac{\delta^2 \mu}{3}}$$

Ex. create a variable for each potential edge

X = total # edges

$p = \frac{1}{n}, \quad \mu = \frac{n(n-1)}{2} \cdot p = \frac{n-1}{2}$

$$\Pr\left[\left|\#\text{edges} - \mu\right| > \frac{n-1}{6} = \frac{1}{3}\mu\right] < 2e^{-\frac{\mu}{27}}$$

$$= 2e^{-\frac{n-1}{54}}$$

for n big, is exceedingly small.

So, actual # edges tightly concentrated around $p\binom{n}{2}$.

Begin with easy things.

For a graph, the clique #, $\omega(G)$

is the size of the largest clique in G:

$$\max\{|S| : \forall u,v \in S,\ (u,v) \in E\}$$

$$\underset{G \leftarrow G(n,p)}{E}\left[\#\, k\text{-cliques in } G\right]$$

$$= \sum_{\substack{S \subset V \\ |S| = 1}} \Pr\left[S \text{ is a clique in } G\right]$$

$$= \binom{n}{k} p^{\binom{k}{2}} \leq \left(n p^{\frac{k-1}{2}}\right)^k$$

So, if $k = (2+\varepsilon)\lg_2 n + 1$, $p = \frac{1}{2}$

$$p^{\frac{k-1}{2}} = \left(\frac{1}{2}\right)^{\left(\frac{2+\varepsilon}{2}\right)\lg_2 n} = n^{-\left(1+\frac{\varepsilon}{2}\right)}$$

and $np^{\frac{k-1}{2}} = n^{-\varepsilon}$

So, $E[\#\ k\text{-cliques}] \le n^{-\varepsilon k} \to 0$

$$\Pr[\exists \text{ a } k\text{-clique}] = \Pr[\#\ k\text{-cliques} \ge 1]$$

$$\le E[\#\ k\text{-cliques}] \le n^{-\varepsilon k} \to 0$$

as $n \to \infty$

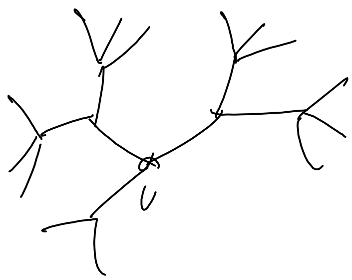
On other hand, for $p = \frac{1}{2}$
$k \ge 2\lg_2 n + 3$

Can show $E[\#\ k\text{-cliques}] > 1$

and, one probably exists
(next lecture)

Tree-like

Graph looks like tree around vertex v (to depth $\frac{k}{2}$)
if is no cycle of length $\leq k+1$
containing v.



Let's show that at most vertizes
looks like tree to depth $\frac{k-1}{2}$ if

$$p < \frac{n^{\theta}}{n} \quad \text{where} \quad \theta < \frac{1-\varepsilon}{k} \qquad \varepsilon > 0$$

Count expected # of k-cycles.

A k-cycle is specified by
first vertex, second, third, ...
$\quad n \qquad\qquad n-1 \qquad\qquad n-2$

but, have over-counted $2k$ times.

So
$$E[\#\ k\text{-cycles}] = \frac{n(n-1)\cdots(n-k+1)}{\;} \, p^k$$

$$2k \qquad p$$

$$\leq \frac{n^k p^k}{2k} \leq \frac{n^{\theta k}}{2k} \leq \frac{n^{-\varepsilon}}{2k}$$

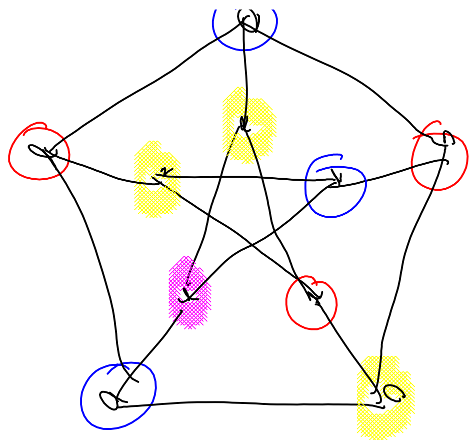
So, Pr[more than $n^{1-\varepsilon}$ verts in k-cycle] $\leq \frac{1}{2}$

or Pr[more than $\frac{n}{2}$ verts in k-cycle] $\leq n^{-\varepsilon}$

Chromatic #, $\chi(G)$

$\chi(G)$ is the least k s.t. the vertices can be divided into k classes with edges only going between classes.

<u>Ex.</u> if Bipartite, $\chi(G) = 2$



$\chi(G) = 4$

Thm [Erdos]

$\forall k, x, \exists$ a graph G

with no k-cycles that

has $\chi(G) \geq x$.

Pf. Consider $p = \frac{n^{\frac{1}{k}+\varepsilon}}{n}$

We showed $\Pr[G \text{ has} > \frac{n}{2}\ k\text{-cycles}] < n^{-\varepsilon}$

If are at most $\frac{n}{2}$ k-cycles,
remove a vertex from each,

to get a graph G' on $\frac{n}{2}$ vertices with no k-cycles.

Would like to say

$$\cancel{\chi(G') \geq \chi(G)},$$

but that's false.

Instead, consider
$\alpha(G)$, the independence # of G

$= \max\{|S| : \forall u,v \in S \ (u,v) \notin E\}$

Have $\alpha(G) \geq \dfrac{n}{\chi(G)}$,
by considering largest class.

We have $\alpha(G') \geq \alpha(G)$

Now $\Pr[\alpha(G) \geq r] \leq E[\text{\# ind sets size } r]$

$$\leq \binom{n}{r}(1-p)^{\binom{r}{2}}$$

$$\leq n^r e^{-\binom{r}{2}p}$$

$$= \left(n e^{-p\left(\frac{r-1}{2}\right)}\right)^r$$

$$-p\left(\frac{r-1}{2}\right)$$

$\to 0$ if $ne \quad \to 0$

For $p \geq \frac{6x \ln n}{n}$, $r \geq \frac{\frac{1}{2}n}{x}$

$$ne^{-\frac{p(r-1)}{2}} = ne^{-\frac{pr}{2} + \frac{p}{2}}$$

$$\leq ne^{-\frac{3}{2}\ln n + p/2}$$

$$\leq n \cdot n^{-\frac{3}{2}} e^{\frac{1}{2}} = \sqrt{e/n}$$

$\to 0$ as $n \to \infty$

If $\alpha(G) \geq \frac{n}{2x}$, $\alpha(G') \geq \frac{n}{2x}$

and # verts $(G') \geq \frac{n}{2}$, so

$\chi(G') \geq x$.

In our case, $p = \frac{n^{\frac{1-\varepsilon}{k}}}{n}$

so, for n suff large,

$n^{\frac{1-\varepsilon}{k}} \geq 6x \ln n$.