11/14: Lecture 20, Planar Graphs

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12:44 PM

There are many equivalent definitions of planar graphs.
We'll start with one of the most restrictive.
We say that a graph G = ({1,..,n},E) is planar if there exist distinct points x_1, …, x_n in the plane such that for all pairs of distinct edges (i,j) and (k,l) in E, the segment from x_i to x_j does not cross the segment from x_k to x_l.

That is, the graph can be drawn in the plane with no crossing edges.

It is equivalent to define planar graphs by allowing a smooth or polygonal curve for each edge. This is not necessarily obvious. We use the first definition as it is a lot easier to work with.

Tutte's Theorem

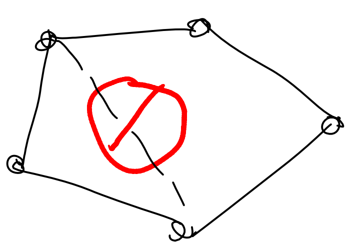
In a beautiful paper, "How to draw a graph" *Proc. London Math. Soc. (3)* **13** (1963) 743-767, Tutte presented a very simple algorithm for computing an embedding of a planar graph in which all the edges were straight lines and didn't cross. This algorithm works provided that the graph is 3-connected. If the graph is not 3-connected, then it is easy to break it into 3-connected components, solve for each of them, and then string the solutions together like beads on a necklace.

First, let's assume that the graph contains a triangle, let's say on vertices 1, 2 and 3. Fix the locations of these vertices to be the points of a regular triangle, say putting vertex i at (cos 2 pi i / 3, sin 2 pi i / 3). We then set every other point to be the average of its neighbors. This results in system of linear equations. It has a unique solution. We already know this! To find the x-coordinate, fix

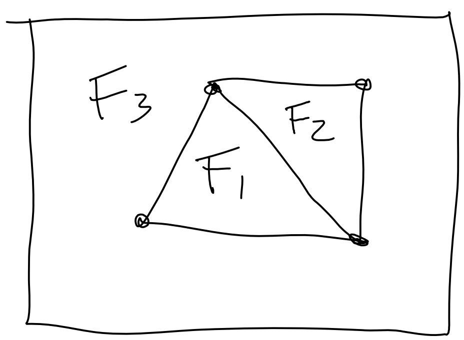
x(i) = cos (2 pi i / 3), for i = 1, 2, 3,

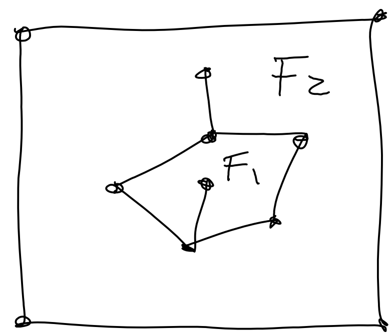
And, then require x(i) to be the average of the values at its neighbors for i > 3. The resulting system of equations exactly corresponds to the problem of computing induced voltages. We can solve for the y-coordinates similarly.

If the graph does not contain a triangle, we can start from any minimal cycle, where by a minimal cycle I mean a cycle for which there are no other edges between the vertices involved. We then locate the vertices of the cycle at the corners of a regular polygon. In fact, would could use the corners of any convex polygon--regularity is unnecessary.



If we inscribe a planar graph in the plane, and then remove all the vertices and edges, we break the plane into a collection of regions. These are called the faces of the graph. Note that there is always at least one face. Here are some examples:





One of the most important facts about planar graphs is Euler's theorem, which says:

Theorem: Let G be a connected planar graph with V vertices, F faces and E edges. Then

$$|V| - |E| + |F| = 2.$$

Let's prove it. First, consider the case in which G is tree. In this case there is exactly one face, so $|F|= 1$, and $|E| = |V| - 1$. So, the equation is satisfied. We now complete the proof by induction on $|E|$, holding v fixed. If G is connected and not a tree, then it contains a cycle. As a cycle divides the plane into two regions, the faces inside the cycle are different from those outside. Let e be any edge on the cycle, and let f1 and f2 be faces on either side of e. Now, remove e. As e was on a cycle, the graph remains connected. But, now the two faces f1 and f2 become one face. The number of vertices is unchanged. As both $|F|$ and $|E|$ has decreased by 1, if the equation is true for the new graph, it is true for the original. So, the theorem follows for fixed $|V|$ by induction on $|E|$.

From this, we can show that planar graphs cannot have too many edges.

Theorem: Let G be a connected planar graph. Then

$$|E| <= 3 |V| - 6.$$

proof. Let e(f) be the number of edges on face f. As each edge lies on two faces (or maybe one face twice), we have

$$|E| = \frac{1}{2}\sum_{f \in F} e(f) \geq \frac{1}{2}\sum_{f \in F} 3 = \frac{3}{2}|F|$$

$$| \ | \leq \underbrace{\quad}_{f \in F} \qquad \qquad f \in F$$

As each face has at least 3 sides. So,

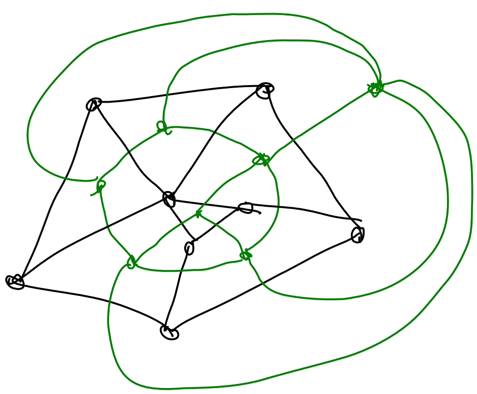
$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E|$$

$$= |V| - \frac{1}{3}|E|,$$

which implies the theorem.

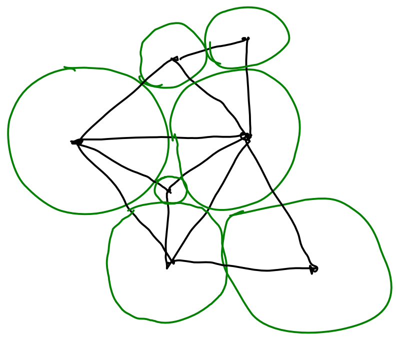
Be warned that, while planar graphs have few edges, they can have low diameter: consider the complete binary tree.

One of the most important concepts about planar graphs is the dual.
The vertices of the dual are the faces, and two vertices in the dual share an edge if and only if the corresponding faces share an edge. Here is an example, with the dual in green.



There are many powerful characterizations of planar graphs. I'll now introduce my favorite by introducing a way of drawing a planar graph, which turns out to be a way of drawing any planar graph.

A circle packing is a collection of circles in the plane whose interiors are disjoint. The graph corresponding to a circle packing has one vertex for each circle, with an edge between each pair of circles that touch. To see that this graph is planar, just locate the vertices at the centers of the circles. Here's an example:



The Koebe-Andreev-Thurston theorem says that every planar graph can be drawn this way. That is, for every planar graph G = (V,E), there is a collection of circles C_1, …, C_n such that C_i and C_j touch if and only if (i,j) is in E.

The proof is topological, and uses a fixed point theorem.

I will now use this characterization to prove a Planar Separator Theorem:

For every planar graph, G = (V,E), there is a set of vertices S such that
$|S| \le 2\sqrt{n}$, and if S is removed
then the remaining graph is disconnected,

and each of its pieces has ~~at~~ most $\frac{3}{4}|V|$ vertices.

The first planar separator theorem was proved by Lipton and Tarjan, and is algorithmic.

The proof that I present here is a simplification by Agarwal and Pach of a theorem of Miller, Thurston, Vavasis and Teng.

See the first 3 pages of the linked paper by myself and Teng for the proof.