Spectra of Graphs

Lecturer: Daniel A. Spielman September 21, 2006

The purpose of this lecture is to examine the spectra of some fundamental graphs. There are many matrices that one can associate with a graph, of which we have seen the adjacency matrix and the walk matrix. A little into this lecture, we will find it convenient to introduce one more matrix: the Laplacian matrix of a graph.

5.1 Spectral Graph Drawing

I'll begin by showing you how, for some graphs, one can learn a lot about them by looking at their eigenvectors. In particular, we will use the eigenvectors to draw the graphs.

We will use right-eigenvectors of the walk matrix. As the eigenvector of eigenvalue 1 is the all-1's vector, it will not be very useful for drawing the graph. So, instead we will consider using other eigenvectors whose eigenvalues are close to 1.

Consider the equation that these eigenvectors satisfy:

$$\mu x = D^{-1} A x.$$

For each vertex i, this equation implies:

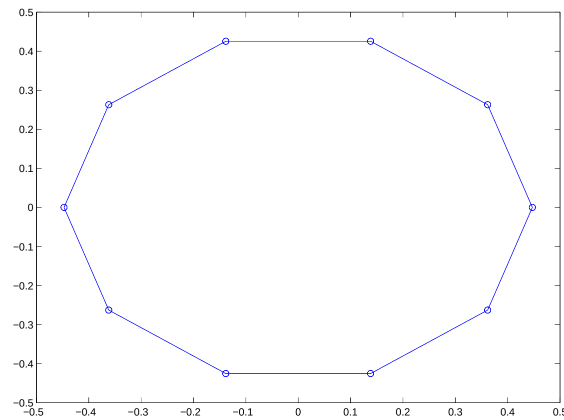
$$\mu x(i) = \frac{1}{d_i} \sum_{(i,j) \in E} x(j).$$

So, when μ is close to 1, the value assigned to $x(i)$ is close to the average value assigned to its neighbors.

Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the eigenvalues of $D^{-1}A$, and let $v_1, \ldots, v_n$ be the corresponding eigenvalues. In the following pictures, we will draw vertex i at coordinate $(v_2(i), v_3(i))$.

Let's begin by generating and drawing the ring graph this way.

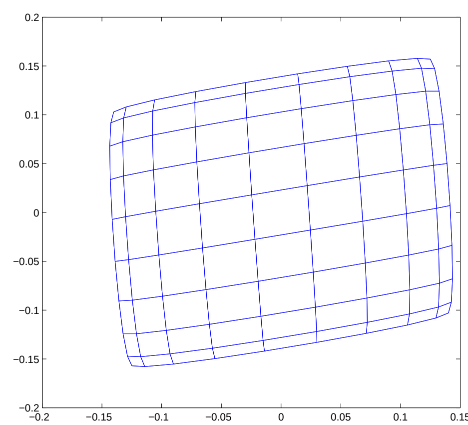
```
a = diag(ones(1,9),1);
a(1,10) = 1;
a = a + a';
s = sum(a);
di = diag(sparse(1./s));
m = di*a;
[v,e] = eig(full(m));
gplot(a,v(:,[8 9]));
hold on
gplot(a,v(:,[8 9]),'o');
```



This is a very good picture of a ring.

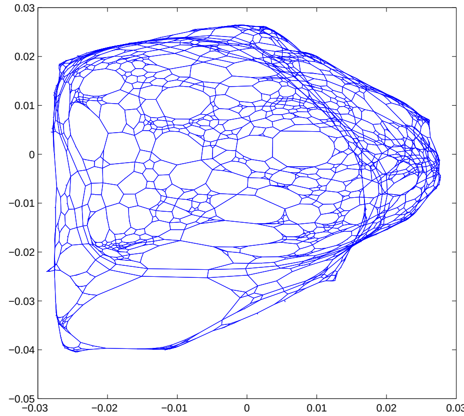
Now, let's try a 10-by-10 grid (in the following 3 and 6 were the indices of the eigenvectors I wanted).

```
a = grid2(10);
s = sum(a);
di = diag(sparse(1./s));
m = di*a;
[v,e] = eigs(m,6);
e
gplot(a,v(:,[3 6]));
```



The next graph is one I found surfing the web. The vertices are roads in Rome, and they have an edge between them if they meet at an intersection.

```
load rome
e
gplot(a,v(:,[2 3]));
```

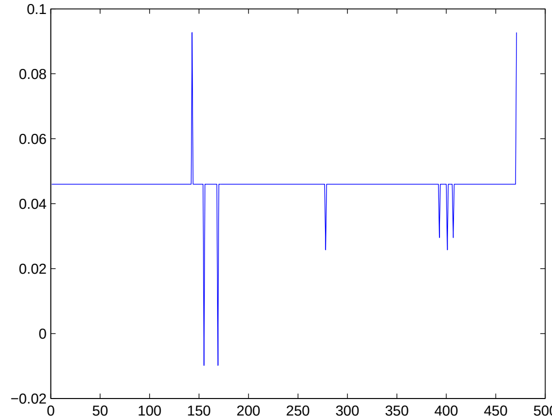


In the next graph, the vertices are everyone who co-authored a paper with Erdos, with edges between each pair who co-authored a paper with each other. Here is the beginning of the transcript of my Matlab session.

```
>> load erdosGraph
>> size(a)

ans =

   471   471

>> s = sum(a);
>> di = diag(sparse(1./s));
>> m = di*a;
>> [v,e] = eigs(m,6);

. . .

>> e

e =

   -1.0000         0         0         0         0         0
         0    1.0000         0         0         0         0
         0         0    1.0000         0         0         0
         0         0         0   -1.0000         0         0
         0         0         0         0   -1.0000         0
         0         0         0         0         0    1.0000
```

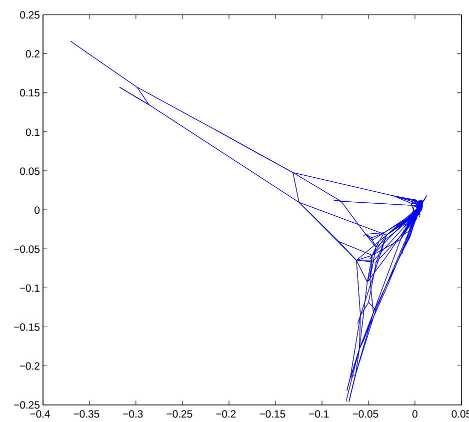
So, this graph has many eigenvalues of 1, and some of -1. This means that it has many connected components. In the following code, I will extract the largest component by looking at an eigenvector of eigenvalue 1. I will do it by grabing the vertices with the most common value. If you try this at home, the most common value might be different.

```
>> plot(v(:,2))
>> ind = find(abs(v(:,2)-.045) < .005);
>> size(ind)

ans =

   463     1
```



Now, let's plot the graph induced on this component.

```
>> b = a(ind,ind);
>> s = sum(b);
>> di = diag(sparse(1./s));
>> m = di*b;
>> [v,e] = eigs(m,6);
>> diag(e)

ans =

    1.0000
    0.9506
    0.9291
    0.9003
    0.8873
    0.8832

>> gplot(b,v(:,2:3))
```



5.2 The Ring Graph

Let R_n denote the ring graph on n vertices. That is, the graph on vertex set $\{1, \ldots, n\}$ with edges between vertices i and $i+1$ for $1 \leq i < n$, and the edge $(1, n)$. Let A be the adjacency matrix of this graph, and let $M = (1/2)A$ be the walk matrix of this graph (it is A divided by 2 because every vertex has degree 2). We will find eigenvectors of M by guessing their identity, and then verifying that our guesses are correct.

To begin, draw this graph in the most-natural way as a regular n-gon, with vertex u at point

$z_u = (\cos\frac{2\pi u}{n}, \sin\frac{2\pi u}{n})$. From this picture, it is geometrically obvious that $(z_u + z_{u+2})/2$ is a multiple of z_{u+1}. This means that the vectors x and y given by

$$x(u) = \cos\frac{2\pi u}{n}, \text{ and}$$
$$y(u) = \sin\frac{2\pi u}{n}$$

are eigenvectors of M. To compute the eigenvalue, let's look at the equation

$$(z_{n-1} + z_1)/2 = \mu z_n.$$

It is clear that the value of μ is the common x-coordinate of z_1 and z_{n-1}, which is $\cos(2\pi/n)$.

Using different pictures of this graph, we can compute the other eigenvalues. For example, if we fix a k and draw u at point $(\cos\frac{2\pi ku}{n}, \sin\cos\frac{2\pi ku}{n})$, then we can similarly argue that we obtain a pair of eigenvectors of eigenvalue $\cos(2\pi k/n)$. So long as $k < n/2$, all the eigenvectors generated this way are distinct. If n is odd, then these yield $n-1$ eigenvectors, which when combined with the all-1's vector gives the desired total of n.

If n is even, then setting $k = n/2$ yields a drawing of the graph that only uses the points $(1, 0)$ and $(-1, 0)$, and so it only provides only one more eigenvector. Combining this with the all-1's eigenvector yields n eigenvectors, in the case that n is even.

Now, let's examine the spectral gap of the ring graph. It is

$$\gamma = 1 - cos(2\pi/n) \to \frac{2\pi^2}{n^2},$$

as n grows large.

This is a very small spectral gap.

5.3 The Path Graph

We let P_n denote the ring graph on n vertices: the graph on vertex set $\{1, \ldots, n\}$ and edges $(i, i+1)$ for $1 \leq i < n$. Let M be the walk-matrix of this graph.

We will show that the right-eigenvectors of M can be obtained from the eigenvectors of the walk-matrix of R_{2n-2}. The idea is to line up each vertex of P_n with one or two vertices of R_{2n-2}: we associate vertex 1 of P_n vertex 1 of R_{2n-2}, vertex n of P_n with vertex n of R_{2n-2}, and vertex i of P_n (for $1 < i < n$) with both vertices i and $2n - i$ of R_{2n-2}. It is not to difficult to see that, under this identification, a random walk on R_{2n-2} becomes a random walk on P_n.

So, any eigenvector w of the walk matrix for R_{2n} that satisfies $w(i) = w(2n - i)$ for $1 < i < n$ becomes an eigenvector of M. So, if we take an eigenvector of the walk matrix of R_{2n} of the form

$$w(u) = \cos\frac{2\pi k(u-1)}{2n},$$

then when we restrict this vector to its first n coordinates, we obtain an eigenvector of M.

5.4 Bounding Eigenvalues, I

There are very few graphs whose eigenvalues and eigenvectors can be analytically derived. Typically, we just try to prove some rough statements about the spectral gap, and the corresponding eigenvectors. The main tool we use to do this is the Courant-Fischer Theorem.

Theorem 5.4.1 (Courant-Fischer). *Let A be a symmetric matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Then, for all $1 \leq k \leq n$,*

$$\mu_k = \max_{S \text{ of dimension } k} \min_{x \in S} \frac{x^T A x}{x^T x}, \quad \textit{and}$$

$$\mu_k = \min_{S \text{ of dimension } n-k+1} \max_{x \in S} \frac{x^T A x}{x^T x}.$$

Note that the term

$$\frac{x^T A x}{x^T x}$$

is usually called the Rayleigh quotient of x. For an eigenvector v, the Rayleigh quotient of v is its eigenvalue.

We often exploit this theorem through the following corollary.

Corollary 5.4.2. *Let A be a symmetric matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and corresponding eigenvectors $v_1, \ldots, v_n$. Then, for all $1 \leq k \leq n$,*

$$\mu_k = \max_{x \perp v_1, \ldots, v_{k-1}} \frac{x^T A x}{x^T x}, \quad \textit{and}$$

$$\mu_k = \min_{x \perp v_{k+1}, \ldots, v_n} \frac{x^T A x}{x^T x}.$$

As an example of how we will use this theorem, we will prove that the second-largest eigenvalue of a walk matrix is small by considering the normalized walk matrix (which is symmetric), and then evidencing a vector orthogonal to the largest eigenvector that has Rayleigh quotient close to 1.

5.5 The Complete Binary Tree

Consider the complete binary tree on $n = 2^k - 1$ vertices. To describe how I will label the vertices, I'll do the example with $n = 15$. I will make vertex 1 the root, vertices 2 through 8 the left subtree with 5 through 8 being the leaves, and vertices 9 through 15 the right sub-tree with 12 through 15 being the leaves. Let D be the diagonal matrix of the degrees, and as in the last lecture let

$$N = D^{-1/2} A D^{-1/2}.$$

Recall from last lecture that the eigenvector of eigenvalue 1 of N is

$$v_1 = (\sqrt{d_1}, \ldots, \sqrt{d_n}),$$

which in our example is

$$(\sqrt{2}, \sqrt{3}, \sqrt{3}, \sqrt{3}, 1, 1, 1, 1, \sqrt{3}, \sqrt{3}, \sqrt{3}, 1, 1, 1, 1).$$

The vector x that we will use to lower bound μ_2 will have the form

$$x = (0, \sqrt{3}, \sqrt{3}, \sqrt{3}, 1, 1, 1, 1, -\sqrt{3}, -\sqrt{3}, -\sqrt{3}, -1, -1, -1, -1).$$

It should be clear that x is orthogonal to v_1, and I hope the general construction of x is clear as well.

Let's begin by computing the denominator of the Rayleigh quotient. Observing that the tree has $(n+1)/2$ leaves, and $(n-3)/2$ internal nodes other than the root, we find

$$x^T x = 2 + 3(n-3)/2 + 1(n+1)/2 = 2n - 2.$$

To compute the numerator, note that

$$\begin{aligned} D^{-1/2} x &= (0, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1), \text{ and} \\ A D^{-1/2} x &= (0, 2, 3, 3, 1, 1, 1, 1, -2, -3, -3, -1, -1, -1, -1). \end{aligned}$$

That is, $AD^{-1/2}x$ is zero at the root, 2 and -2 at the children of the root, 3 or -3 at all other internal nodes, and 1 or -1 at all leaves. So,

$$x^T D^{-1/2} A D^{-1/2} x = 2 \cdot 2 + 3(n-7)/2 + (n+1)/2 = 2n - 6.$$

So,

$$\frac{x^T N x}{x^T x} = \frac{2n-6}{2n-2} = 1 - \frac{2}{n-1}.$$

So, we learn that the spectral gap of the complete binary tree is at most $2/(n-1)$.

Still, that computation was not as much fun as it could have been. So, we'll derive a simpler way of bounding spectral gaps.

5.6 Comparison with Laplacians

To focus on the spectral gap, it is convenient to shift the eigenvalues of N so that the gap occurs near zero. To this end, we consider the matrix

$$\mathcal{L} \stackrel{\text{def}}{=} I - N = D^{-1/2}(D - A)D^{-1/2},$$

which is called the normalized Laplacian matrix. The eigenvalues of $\mathcal{L}$ will be denoted $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and satisfy

$$\lambda_i = 1 - \mu_i,$$

where μ_i is the ith largest eigenvalue of N.

We will exploit the fact that the normalize Laplacian is a scaling of an even simpler matrix: the Laplacian of a graph which is given by

$$L = D - A.$$

We will show that one can use a bound on the smallest non-zero eigenvalue of L to obtain a bound on the smallest non-zero eigenvalue of $\mathcal{L}$, which will be very helpful as the eigenvalues of Laplacians and normalized Laplacians are particularly easy to bound. The reason is that

$$x^T L x = \sum_{(i,j) \in E} (x(i) - x(j))^2. \tag{5.1}$$

To exploit this formula to bound the second-smallest eigenvalue of the normalized Laplacian, note that

$$\begin{aligned} \lambda_2 &= \min_{S \text{ of dim } 2} \max_{x \in S} \frac{x^T D^{-1/2} L D^{-1/2} x}{x^T x} \\ &= \min_{S \text{ of dim } 2} \max_{y \in S} \frac{y^T L y}{y^T D y}, \end{aligned}$$

by setting $y = x D^{-1/2}$.

So, to upper bound λ_2 for the complete binary tree, it suffices to take S to be the span of the all-1's vector and the vector that is 0 at the root, 1 on the right sub-tree, and -1 on the left subtree. Call this later vector x. So, any vector in this vector space can be written as a multiple of $\alpha \mathbf{1} + x$. We then have

$$(\alpha \mathbf{1} + x)^T L (\alpha \mathbf{1} + x) = x^T L x = 2,$$

as adding a constant α to each vertex does not change expression (5.1). By a little calculus, one can show that

$$(\alpha \mathbf{1} + x)^T D (\alpha \mathbf{1} + x) \geq x^T D x,$$

which is the sum of the degrees of the vertices other than the root.

5.7 Planar Graphs

For example, we can also use this argument to get a bound on the spectral gap of the n-by-n grid. If n is even, we create a vector that assigns 1 to the left-half and -1 to the right half. We will then have $x^T L x = 4n$, and $x^T x$ equal to the sum of the degrees of vertices in the graph, which is just a little under $4n^2$. So, we get an upper bound on the second-smallest eigenvalue of the normalized Laplacian that is just a little larger than $1/n$.

To describe a better test vector, let me describe the vertices of the grid as pairs (i, j). Then, if we set $x(i, j) = i - n/2$, we get a stronger bound. A little arithmetic shows that the Rayleigh quotient of this vector is some constant times $1/n^2$. This is almost tight.

In fact, one can prove similar bounds for all planar graphs. Teng and I proved that for every planar graph, the second-smallest eigenvalue of the Laplacian satisfies $\lambda_2 \leq 8 d_{max}/n$, where d_{max} is the largest degree of a vertex in the graph, and n is the number of vertices.