weighted edge $\Rightarrow$ resistor

$$\text{Resistance} = \frac{1}{\text{weight}}$$

$$\text{Conductance} = \text{weight}$$

**Ohm's Law:** $i(x,y) =$ current flow from $x \rightarrow y$ satisfies

$$i(x,y) = \frac{(V(x) - V(y))}{C_{x,y}} = \frac{(V(x) - V(y))C_{x,y}}{C_{x,y}}$$

so, $i(x,y) = -i(y,x)$

**Kirchhoff's Current Law:**

if node $x$ not connected to battery,

total current flow out of $x = 0$

i.e.

$$\sum_{y \sim x} i(x,y) = 0$$

Combining with Ohm's Law, says

$$0 = \sum_{y \sim x} i(x,y) = \sum_{y \sim x} \left(\frac{(V(x) - V(y))C_{x,y}}{C_{x,y}} \right)$$

$$0 = \frac{dx}{C_{x}} V(x) - \sum_{y \sim x} C_{x,y} V(y)$$

so, $V(x) = \frac{1}{dx} \sum_{y \sim x} C_{x,y} V(y)$, where $dx = \sum_{y \sim x} C_{x,y}$
$V(k)$ is a weighted average of $v(k)$

Gives one equation for each non-terminal vertex.

Ex. 

\[
\begin{array}{ccc}
1 & \frac{1}{2} & 0 \\
\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\
\sqrt{\frac{1}{4}} & \sqrt{\frac{1}{4}} & 0 \\
\end{array}
\]

Def \( f : V \rightarrow \mathbb{R} \) is harmonic on \( W \subseteq V \) if \( f(u) \) we W

\[
f(u) = \frac{1}{d_w} \sum_{w \in w} C_{uw} f(w)
\]

Lem: If \( G \) is connected, \( W \subseteq V \), \( W \neq V \) and \( f \) and \( g \) are harmonic on \( W \), and \( V \setminus W \) \( f(V) = f(g) \) then \( f = g \)

pf: Consider \( h = f - g \). \( h \) is harmonic on \( W \). \( h(t) = 0 \) \( \forall t \in \partial W \).
Now, let \( h(z) \) be maximum.

Then \( h(z) = \sum_{y \in z} \frac{C_{yz}}{d_{z}} \cdot h(y) \)

as
1. \( h(y) \leq h(z) \)
2. \( \sum_{y \in z} \frac{C_{yz}}{d_{z}} = 1 \)
3. \( C_{yz} > 0 \)

we know \( h(y) = h(z) \)

So, by induction, for all \( y \) reachable from \( z \), \( h(y) = h(z) \)

Including \( y \in V - W \Rightarrow h(z) = 0 \)

Similarly, can show \( \min_{z} h(z) = 0 \).

So, \( h = 0 \).

This implies the solution to a harmonic system is unique.

But, do voltages exist?

Yes.

Given a graph, nodes \( s \) and \( t \), want \( f: V \rightarrow \mathbb{R}^+ \)
\( f(s) = 1 \)
\[ f(A) = 0 \]  
\[ f \text{ harmonic on } U - \{s, t\} \]

Try

\[ F(x) = \Pr \left[ A \text{ and walk from } x \text{ hits } s \text{ before } t \right] . \]

Clearly, \( F(s) = 1 \), \( F(t) = 0 \), and

\[ F(x) = \sum_{y \sim x} \Pr[\text{first step from } x \to y] F(y) \]

\[ = \sum_{y \sim x} \frac{c_{xy}}{c_x} F(y) \]

So, \( F \) satisfies the equations and by uniqueness, \( F = U \).

The effective conductance between \( s \) and \( t \)

is the total current flow when \( U(s) = 1, U(t) = 0 \)

Let's check its well-defined. Namely, that

\[ \sum_{x \sim s} \frac{\xi}{c_x} (s, x) = \sum_{x \sim t} \frac{\xi}{c_x} (x, t) \]
\[ O = \sum_{x} \sum_{y \sim x} i(x, y) \quad (\text{as } i(x, y) = -i(y, x)) \]

\[ = \sum_{y \sim s} i(s, y) + \sum_{y \sim t} i(t, y) + \sum_{x \in U - \{s, t\}} \sum_{y} i(x, y) \]

\[ = 0 \]

so, \( O = \sum_{y \sim s} i(s, y) + \sum_{y \sim t} i(t, y) \) \( \checkmark \)

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What is chance hit \( t \) before return to \( s \)?

Denoted \( P_{s \rightarrow t} \), also called escape probability

\[ = \sum_{y \sim s} \Pr[\text{first step to } y] \cdot (1 - F(y)) \]

\[ = \sum_{y \sim s} \frac{c_{s, y}}{ds} (U(s) - U(y)) = \frac{1}{ds} \sum_{y \sim s} i(s, y) = \frac{1}{ds} \text{Ceff} \]

From this, can show

\[ E[\# \text{times return to } s \text{ before hit } t] = ds \text{ Reff} \]

where \( \text{Reff} = \frac{1}{\text{Ceff}} \) is effective resistance

Generally, can show
Finally, if flow $I$ from $s$ to $t$, voltages $V$ satisfy

$$LV = (X_s - X_t)$$

so, $V = L^+ (X_s - X_t)$

and, for any $x$ and $y$, potential diff between $x$ and $y$ is $V(x) - V(y)$

$$= (X_x - X_y)^T L^+ (X_s - X_t)$$

$\Rightarrow$ Pot diff between $x$ and $y$ when flow $I$ current from $s$ to $t$

$$= Pot diff between s and t when flow $I$ current from $x$ to $y$