9/28 - Resistor Networks

Friday, September 29, 2006
1:37 PM

weighted edge → resistor

resistance = $\frac{1}{\text{weight}}$

conductance = weight

Ohm's Law: $i(x,y)$ = current flow from x to y satisfies

$$i(x,y) = \frac{(V(x) - V(y))}{r_{x,y}} = (V(x) - V(y))\, C_{x,y}$$

so, $i(x,y) = -i(y,x)$

Kirchoff's Current law:

if node x not connected to battery,
total current flow out of $x = 0$

i.e. $\displaystyle\sum_{y \sim x} i(x,y) = 0$

Combining with Ohm's law, says

$$0 = \sum_{y \sim x} i(x,y) = \sum_{y \sim x} (V(x) - V(y))\, C_{x,y}$$

$$= d_x V(x) - \sum_{y \sim x} C_{x,y} V(y)$$

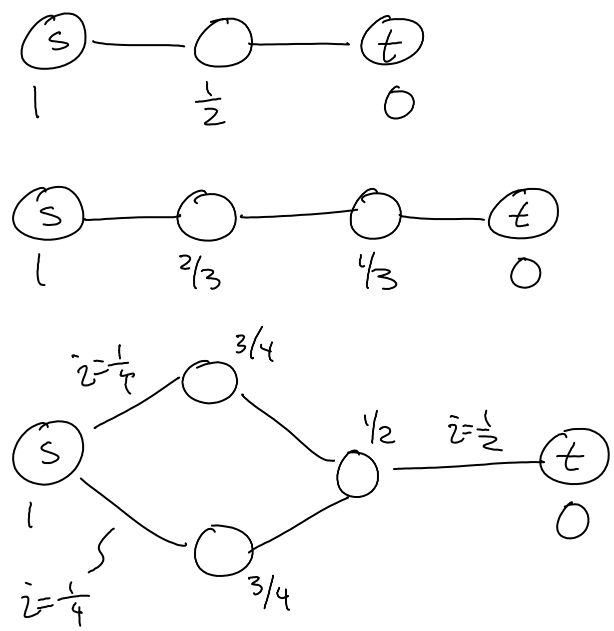
so, $V(x) = \frac{1}{d_x} \sum C_{x,y} V(y)$ where $d_x = \sum C_{x,y}$

$\qquad\qquad y \sim x \qquad\qquad\qquad\qquad y \sim x$

$V(x)$ is a weighted average of nbrs

Gives one equation for each non-terminal vertex.

Ex.



<u>Def</u> $f : V \to \mathbb{R}$ is <u>harmonic</u> on $W \subseteq V$
if $\forall w \in W$

$$f(w) = \frac{1}{d_w} \sum_{y \sim w} c_{w,y} f(y)$$

<u>Lem</u> If G is connected, $W \subset V$, $W \neq V$
and f and g are harmonic on W,
and $\forall x \notin W$ $f(x) = f(g)$
then $f = g$

<u>pf.</u> Consider $h = f - g$.
Is harmonic on W.
$h(x) = 0 \quad \forall x \notin W$.

Now, let $h(z)$ be maximum.

Then $h(z) = \sum_{y \sim z} \frac{c_{zy}}{d_z} h(y)$

as 1. $h(y) \leq h(z)$

2. $\sum_{y \sim z} \frac{c_{zy}}{d_z} = 1$

3. $c_{zy} > 0$

we know $h(y) = h(z)$

So, by induction, for all y reachable from z, $h(y) = h(z)$

Including $y \in V - W \Rightarrow h(z) = 0$

Similarly, can show $\min_z h(z) = 0$.

So, $h = 0$.

This implies the solution to a harmonic system is unique.

But, do voltages exist?

Yes.

Given a graph, nodes s and t, want a $f: V \to \mathbb{R}$ s.t.

$f(s) = 1$

$f(t) = 0$
f harmoniz on $V - \{s, t\}$

Try

$F(x) = \Pr$ [a rand walk from x hits s before t].

Clearly, $F(s) = 1$, $F(t) = 0$, and

$$F(x) = \sum_{y \sim x} \Pr[\text{first step from } x \text{ to } y]\, F(y)$$

$$= \sum_{y \sim x} \frac{c_{xy}}{d_x} F(y)$$

So, F satisfies the equations and by uniqueness, $F = V$.

The effective Conductance between s and t is the total current flow when $V(s) = 1$, $V(t) = 0$

Let's check its well defined. Namely, that

$$\sum_{x \sim s} i(s, x) = \sum_{x \sim t} i(x, t)$$

pf.

$$0 = \sum_{x} \sum_{y \sim x} i(x,y) \qquad (\text{as } i(x,y) = -i(y,x))$$

$$= \sum_{y \sim s} i(s,y) + \sum_{y \sim t} i(t,y) + \underbrace{\sum_{x \in V - \{s,t\}} \sum_{y} i(x,y)}_{0}$$

$$\text{so, } 0 = \sum_{y \sim s} i(s,y) + \sum_{y \sim t} i(t,y) \quad \checkmark$$

What is chance hit t before return to s?
Denoted $\Pr[s \to t]$, also called
escape probability

$$= \sum_{y \sim s} \Pr[\text{first step to } y] \cdot (1 - F(y))$$

$$= \sum_{y \sim s} \frac{c_{s,y}}{d_s} (V(s) - V(y)) = \frac{1}{d_s} \sum_{y \sim s} i(s,y) = \frac{1}{d_s} C_{\text{eff}}$$

From this, can show

$$E[\#\text{ times return to } s \text{ before hit } t] = d_s R_{\text{eff}}$$

where $R_{\text{eff}} = \frac{1}{C_{\text{eff}}}$ is effective resistance

Generally, can show

E[# times a walk starting at s hits x before t]

$$= d_x R_{eff} V(x)$$

Finally, if flow 1 from s to t,
Voltages v satisfy

$$L v = (\chi_s - \chi_t)$$

So, $v = L^+(\chi_s - \chi_t)$

and, for any x and y, potential diff between x and y is $v(x) - v(y)$

$$= (\chi_x - \chi_y)^T L^+ (\chi_s - \chi_t)$$

$\Rightarrow$ Pot diff between x and y when flow 1 current from s to t

= Pot diff between s and t when flow 1 current from x to y