Graphs and Networks **out:** September 26, 2006

Problem Set 1-Ugrad

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1 Homework Policy

You may discuss the problems with other students. But, you must write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

You **may not** search the web for solutions to similar problems given out in other classes. If you think this policy needs any clarification, please let me know.

1. In the following problems, $G = (V, E)$ is an undirected, connected graph. Let d_i be the degree of vertex i, let D be the diagonal matrix with diagonal $d_1, \ldots, d_n$, and let A be the adjacency matrix of G. Let $M = D^{-1}A$.

   (a) Prove that the only vectors v satisfying

   $$Mv = v$$

   are constant vectors. Hint: Modify the proof of Lemma 4.3.1 from Lecture 4.

   (b) Prove that if G is bipartite, then there exists a vector v satisfying

   $$Mv = -v.$$

   Hint: Modify the a vector from $(1a)$ using the bipartition. If you are stuck, first consider the example of the graph with two vertices and one edge.

   (c) Prove that if G is connected and there exists a vector v satisfying

   $$Mv = -v,$$

   then G is bipartite.
   Hint: Modify the proof of $(1a)$.

2. Let $G = (V, E)$ be a directed, strongly connected graph. As in Lecture 2, we let d_i^+ be the out-degree of vertex i, let D be the diagonal matrix with diagonal $d_1^+, \ldots, d_n^+$, let A be the adjacency matrix of G, and let $M = D^{-1}A$. We will now prove that if r is a vector that satisfies

   $$rM = r, \quad \text{and} \quad r(1) > 0,$$

then r must be non-negative.

Our proof will use the following matrix:

$$M^* \stackrel{\text{def}}{=} (1/n) \sum_{i=1}^{n} M^i.$$

(a) Prove that if $r = rM$, then $r = rM^*$.

(b) Prove that $M^* \mathbf{1} = \mathbf{1}$.

(c) Prove that M^* has no negative or zero entries (this is the interesting part).

We may now conclude that a vector r satisfying $r = rM$ and $r(1) > 0$ must be non-negative by observing that this implies:

$$\forall j, r(j) = \sum_i r(i) M^*_{i,j},$$

which implies that

$$\forall j, |r(j)| \leq \sum_i |r(i)| \, M^*_{i,j}, \tag{1}$$

and so

$$\begin{aligned} \sum_j |r(j)| &\leq \sum_j \sum_i |r(i)| \, M^*_{i,j} \\ &= \sum_i |r(i)| \sum_j M^*_{i,j} \\ &= \sum_i |r(i)| \, , \end{aligned}$$

as $M^* \mathbf{1} = \mathbf{1}$ implies $\sum_j M^*_{i,j} = 1$. Note that if the "$\leq$" in the inequality above was a "$<$", then we would have a contradiction.

(d) Prove that if r has negative entries, then we obtain such a contradiction by showing that the "$\leq$" in the inequality in (1) must be a "$<$".

3. Let G be an undirected, connected graph. Let $L = D - A$ be its Laplacian matrix, and let $\mathcal{L} = D^{-1/2}(D - A)D^{-1/2}$ be its normalized Laplacian matrix. Let λ_2 be the second-smallest eigenvalue of L, and let $\tilde{\lambda}_2$ be the second-smallest eigenvalue of $\mathcal{L}$. Prove that

$$\tilde{\lambda}_2 \leq \lambda_2.$$

Hint: use the Courant-Fischer Theorem.