Graphs and Networks **out:** September 27, 2006

Problem Set 1-Grad

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1 Homework Policy

You are encouraged to collaborate on the problem set. But, you should write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

1. The Perron-Frobenius theorem tell us that if A is the adjacency matrix of a weighted, strongly connected, directed graph, then A has a strictly positive eigenvector v with positive eigenvalue μ_1. Moreover, all other eigenvalues μ_i of A satisfy $|\mu_i| < \mu_1$. It turns out that this theorem is easy to prove in the case that A is stochastic, that is $A\mathbf{1} = \mathbf{1}$. We did most of this proof in class, and the rest appears in the undergraduate problem set.

   In this problem, we will show how to reduce to the case that $A\mathbf{1} = c\mathbf{1}$, at least when A has no zero entries. That is, we will prove that every such A is similar to a constant times a stochastic matrix.

   We will do this algorithmically. Let $A_0 = A$. Then, set $s^{(i)} = A_i\mathbf{1}$, and $D_i = \mathtt{diag}(s^{(i)})$. We then set $A_{i+1} = D_i^{-1}A_iD_i$. We will show that the sequence of matrices A_i is converging to a constant times a stochastic matrix.

   (a) Let $s^{(i)}_{max} = \max_i s^{(i)}$ denote the maximum row-sum in A_i, and $s^{(i)}_{min}$ denote the minimum row sum. Prove that
   $$s^{i+1}_{max} - s^{i+1}_{min} \leq s^{i}_{max} - s^{i}_{min}.$$

   (b) For a matrix A, let $\min(A)$ denote the minimum entry of A. Set
   $$\gamma_i = \frac{\min(A_i)}{s^{(i)}_{max}}.$$
   Prove that
   $$s^{i+1}_{max} - s^{i+1}_{min} \leq (1-\gamma_i)\left(s^{i}_{max} - s^{i}_{min}\right).$$

   (c) Prove that for every matrix A, there exists a constant ϵ such that $\gamma_i \geq \epsilon$, for all i.

   Taken together, these statements show that if A has no zero entries, then the sequence A_i approaches a multiple of a stochastic matrix. With a little analysis, one can extend this to show that A is similar to a multiple of a stochastic matrix.

(extra credit) Extend this analysis to the case in which A can have zero entries, but is the weighted adjacency matrix of a connected graph.

2. We will now take another approach to proving the Perron-Frobenius Theorem. We will use Brouwer's Fixed Point Theorem, which tells us that if S is a closed convex set and f is a continuous function from S into itself, then there exists an $x \in S$ such that $f(x) = x$.

   (a) Prove that if A is non-negative matrix, then there exists a non-negative vector v such that $Av = \lambda v$, for some λ. Hint: apply Brouwer's fixed point theorem to the closed convex set $S = \{v \in \mathbb{R}^n : \sum v_i = 1 \text{ and } v_i \geq 0, \forall i\}$.

   (b) Prove that if A is the adjacency matrix of a weighted, directed, strongly connected graph, then this v must be strictly positive.

   Note that we already proved in class that this implies that v is the only non-negative eigenvector of A.

   (c) Prove that if $uA = \mu u$, then $|\mu| \leq \lambda$. Hint: consider the vector $(|u_1|, \ldots, |u_n|)$, and note that u could have complex entries.

3. Prove that there exists a constannt c such that if L is the Laplacian matrix of the complete binary tree on n vertices, then $\lambda_2(L) \geq 1/cn$. To start, I suggest reading the notes from Lecture 3 of my class "Spectral Graph Theory", which is available on my web page.