Graphs and Networks **out:** October 11, 2006

Problem Set 2-Ugrad

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1 Revisions

- On Oct 16th, changed "m" to "ℓ " in problem 5.
- On Oct 16th, changed "k^2" to "$k(k-1)$" in problem 2b.
- On Oct 16th, second round, $i \leq j$ to $i < j$ in the description of the graph in problem 2.
- On Oct 16th, second round, changed status of problem 5 to "extra credit", and edited the problem some.

2 Homework Policy

You may discuss the problems with other students. But, you must write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

You **may not** search the web for solutions to similar problems given out in other classes. If you think this policy needs any clarification, please let me know.

1. Let $G = (V, E)$ be an undirected weighted graph. Consider the corresponding random walk. Let $x_1, \ldots, x_k$ be a sequence of vertices. Let p_1 be the probability that a random walk starting at x_1 first steps to x_2, then to x_3, and so on until x_k and then returns to x_1 on the kth step. Similarly, let p_i be the probability that a random walk starting at x_i traverses exactly the edges

   $$(x_i, x_{i+1}), (x_{i+1}, x_{i+2}), \ldots, (x_{k-1}, x_k)(x_k, x_1), (x_1, x_2), \ldots, (x_{i-1}, x_i)$$

   in its first k steps.

   Prove that $p_1 = p_2 = \cdots = p_k$.

2. Let G be the graph with vertex set $\{1, 2, \ldots, 2k\}$ and edge set

   $$\{(i,j) : 1 \leq i < j \leq k\} \cup \{(i,j) : k+1 \leq i < j \leq 2k\} \cup \{(1, k+1)\}.$$

   We call this the dumbell graph on $2k$ vertices. Let L be the Laplacian matrix of this graph, and let $\mathcal{L}$ be the normalized Laplacian.

(a.) Prove that $\lambda_2(L) \leq 2/k$.

(b.) Prove that $\lambda_2(\mathcal{L}) \leq 2/(k(k-1))$.

3. Let $G = (V, E)$ be a connected, unweighted, undirected graph with n vertices. Let s and t be distinct vertices in V. Prove that the effective resistance between s and t is at most n.

   Hint: First, consider the case in which the graph just consists of a path from s to t. Then, use Rayleigh's Monotonicity Theorem.

4. Let $G = (V, E)$ be an unweighted, undirected graph. Let x, y and z be distinct vertices of G such that $\{(x, y), (y, z), (x, z)\} \subseteq E$. Also, let s and t be two other vertices of G.

   Let $G' = (V', E')$ be the graph with vertex set $V' = V \cup \{w\}$, where $w \notin V$ and edge set

   $$E' = E - \{(x, y), (y, z), (x, z)\} \cup \{(w, x), (w, y), (w, z)\},$$

   where every edge in E' has weight 1, except those edges attached to w which all have weight 3.

   Prove that the effective resistance in G between s and t is the same as the effective resistance in G' between s and t.

   Hint: Consider the voltages induced in G when $v(s) = 1$ and $v(t) = 0$. How do the induced voltages in in G' compare?

5. **[extra credit] Gossiping with an adversarial scheduler** In this problem, we are going to perform an analysis of the distributed averaging algorithm under the assumption that the communications that occur are being maliciously chosen, as opposed to occuring at random. In particular, we assume that a schedule of communications has been chosen in advance. This schedule consists of a sequence of pairs of vertices, $\left((u_i, v_i)\right)_i$, which indicates that in the ith step vertex u_i communicates with vertex v_i. When vertex u_i communicates with v_i, they average their values. In particular, if x_0 denotes the initial vector of values and x_i denotes the vector present at time i, then

   $$x_{i+1}(u_i) = \frac{x_i(u_i) + x_i(v_i)}{2}, \text{ and}$$
   $$x_{i+1}(v_i) = \frac{x_i(u_i) + x_i(v_i)}{2}.$$

   Of course, if the schedule does not contain a set of edges that results in a connected graph, then the process need not converge. So, we will assume that there is some fixed number $\ell \geq n$ such that for every j, the graph

   $$G = (\{1, \ldots, n\}, \{(u_i, v_i) : j \leq i \leq j + \ell\})$$

   is connected (n is the number of vertices in the graph).

   (a) Let M_j be the matrix such that

   $$x_{j+\ell n} = M_j x_j.$$

   Prove that every entry of M_j is at least $2^{-\ell n}$.

Hint: Let M_j^i denote the matrix such that $x_{j+\ell i} = M_j^i x_j$. For each u, consider the set of v such that $M_j^i(u,v) > 0$. Prove that this set has at least i elements.

(Note, if you can do part (b) without this part, you will get credit for this part too.)

(b) Consider the potential function $f(x) = \max_i x(i) - \min_i x(i)$. Prove that, for all j,

$$f(x_{j+\ell}n) \leq (1 - 2^{-\ell n}) f(x_j).$$