Graphs and Networks **out:** October 11, 2006

Problem Set 2-Grad

Lecturer: Daniel A. Spielman **due:** October 24, 2006

1 Revisions

- On Oct 16th, changed "m" to "ℓ " in problem 4.
- On Oct 16th, changed "k^2" to "$k(k-1)$" in problem 1b.
- On Oct 16th, second round, $i \leq j$ to $i < j$ in the description of the graph in problem 2.
- On Oct 16th, second round, fixed two errors in the Sherman-Morrison formula.
- On Oct 16th, second round, changed ℓ to ℓn many times in problem 4.
- On Oct 16th, third round, fixed another error in the Sherman-Morrison formula.

2 Homework Policy

You are encouraged to collaborate on the problem set. But, you should write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

1. Let G be the graph with vertex set $\{1, 2, \ldots, 2k\}$ and edge set

$$\{(i,j) : 1 \leq i < j \leq k\} \cup \{(i,j) : k+1 \leq i < j \leq 2k\} \cup \{(1, k+1)\}.$$

We call this the dumbell graph on $2k$ vertices. Let L be the Laplacian matrix of this graph, and let $\mathcal{L}$ be the normalized Laplacian.

(a.) Prove that $\lambda_2(L) \leq 2/k$.

(b.) Prove that $\lambda_2(\mathcal{L}) \leq 2/(k(k-1))$.

(c.) Prove that there is a constant c such that for all k, $\lambda_2(L) \geq c/k$.
Hint: There are two ways to approach this problem. The first is to apply the technique used in problem 3 of problem set 1. The other is to observe that the vertices can be partitioned into 4 classes.

2. Let $G = (V, E)$ be a connected, unweighted, undirected graph with n vertices. Let s and t be distinct vertices in V. Prove that the effective resistance between s and t is at most n.

   Hint: First, consider the case in which the graph just consists of a path from s to t. Then, use Rayleigh's Monotonicity Theorem.

3. The Sherman-Morrison Formula from linear algebra tells us that if A is a non-degenerate matrix and u is a column vector, then

$$\left(A + uu^T\right)^{-1} = A^{-1} - \frac{A^{-1}uu^T A^{-1}}{1 + u^T A^{-1} u}.$$

   If A is symmetric and degenerate, and if u is in the span of A, then the same identity holds with the inverse replaced by the pseudo-inverse (you might want to check this).

   Use the Sherman-Morrison Formula to prove Rayleigh's Monotonicity Theorem.

   Hint: recall that if L is the Laplacian of a graph, then $L^+(\chi_s - \chi_t)$ is the vector of potentials that result in a flow of one from s to t.

4. **Gossiping with an adversarial scheduler**

   In this problem, we are going to perform an analysis of the distributed averaging algorithm under the assumption that the communications that occur are being maliciously chosen, as opposed to occuring at random. In particular, we assume that a schedule of communications has been chosen in advance. This schedule consists of a sequence of pairs of vertices, $\left((u_i, v_i)\right)_i$, which indicates that in the ith step vertex u_i communicates with vertex v_i. When vertex u_i communicates with v_i, they average their values. In particular, if x_0 denotes the initial vector of values and x_i denotes the vector present at time i, then

$$x_{i+1}(u_i) = \frac{x_i(u_i) + x_i(v_i)}{2}, \text{ and}$$
$$x_{i+1}(v_i) = \frac{x_i(u_i) + x_i(v_i)}{2}.$$

   Of course, if the schedule does not contain a set of edges that results in a connected graph, then the process need not converge. So, we will assume that there is some fixed number $\ell \geq n$ such that for every j, the graph

$$G = \left(\{1, \dots, n\}, \{(u_i, v_i) : j \leq i \leq j + \ell\}\right)$$

   is connected (n is the number of vertices in the graph).

   (a) Let M_j be the matrix such that
$$x_{j+\ell n} = M_j x_j.$$
   Prove that every entry of M_j is at least $2^{-\ell n}$.
   Hint: Let M_j^i denote the matrix such that $x_{j+\ell i} = M_j^i x_j$. For each u, consider the set of v such that $M_j^i(u, v) > 0$. Prove that this set has at least i elements.
   (Note, if you can do part (b) without this part, you will get credit for this part too.)

   (b) Consider the potential function $f(x) = \max_i x(i) - \min_i x(i)$. Prove that, for all j,
$$f(x_{j+\ell}n) \leq (1 - 2^{-\ell n}) f(x_j).$$