Graphs and Networks **out:** October 25, 2006

Problem Set 3-Grad

Lecturer: Daniel A. Spielman **due:** November 7, 2006

1 Homework Policy

You are encouraged to collaborate on the problem set. But, you should write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

2 Corrections Made

1. (10/27/06) Changed "a leaf" to "any leaf" in the second sentence of problem 1.
2. (10/27/06) Earlier I wrote that the FKG inequality dealt with "indicator functions", whereas I should have said "increasing functions". The correction has been made.
3. (10/27/06) In problem 3, I made it clear that in part (b) you should just deal with one k, and in part (d) forgot to divide $(1+\epsilon)\ln n$ by n. Both corrections have been made.
4. (10/27/06) In problem 4, I've added a statement in parts (a) and (b) saying that you just need to do these for n large.

3 The Problems

1. **Percolation on k-ary Trees**

   Consider the percolation problem on infinite k-ary trees with probability p of keeping each edge. Let $x_d(p)$ denote the probability that the root is connected to any leaf in the k-ary tree of depth d. In the base case, $d = 0$, the tree consists of a single vertex, and by assumption $x_0(p) = 1$.

   (a) Write an expression for $x_{d+1}(p)$ in terms of $x_d(p)$. In particular, find a function $f(x, p)$ such that
   $$x_{d+1}(p) = f(x_d(p), p).$$

   (b) Show that for every ϵ there exists a constant c_ϵ such that if $p = (1+\epsilon)/k$ and $x < c_\epsilon$, then
   $$f(x, p) > x.$$
   You constant c_ϵ should be a function of ϵ alone, and in particular not depend on k.

It is easy to show that $x_{d+1}(p) \leq x_d(p)$ for all d. So, if you've proved part (b), you may now conclude that $x_d \geq c_\epsilon$ for all d.

2. **FKG Inequality**

   Let $T_d^k(p)$ be the distribution on graphs obtained by keeping each edge of the depth-d complete k-ary tree with probability p.

   An increasing function on the space of graphs is a function $f(G)$ such that if H is a subgraph of G, then $f(H) \leq f(G)$. The FKG Inequality says that if f and g are increasing functions then

   $$\mathbf{E}_{G \leftarrow T_d^k(p)}\left[f(G)g(G)\right] \geq \mathbf{E}_{G \leftarrow T_d^k(p)}\left[f(G)\right]\mathbf{E}_{G \leftarrow T_d^k(p)}\left[g(G)\right].$$

   For $G \leftarrow T_d^k(p)$, let $A(G)$ be the event that G contains a path from the root to a leaf. Let $B(G)$ be the event that the root is connected to all of its k children.

   Prove that for $p = (1+\epsilon)k$, there exists a constant c, independent of d but possibly depending on k and ϵ, such that

   $$\mathrm{P}_{G \leftarrow T_d^k(p)}\left[A(G) \text{ and } B(G)\right] \geq c.$$

   Hint: If you use the result of the first problem, this problem is easy.

3. **Threshold for connectivity**

   Consider the random graph model $G(n,p)$, where $p = 2\ln n/n$. We will prove that a graph chosen from this distribution is almost certainly connected. (Hint: this problem does not require any technique more sophisticated than the union bound)

   (a) Prove that it is unlikely that there is any vertex with no neighbors.

   (b) The graph is disconnected if and only if there exists a subset of the vertices $\emptyset \subset S \subset V$ such that G contains no edges between S and $V - S$. Prove for each $0 < k < n$ that it is unlikely there is any set S of size k such that G contains no edges between S and $V - S$. (that is, just prove it for each particular k)

   (c) Prove that it is unlikely that a graph chosen from the distribution $G(n,p)$ is disconnected. (that is, sum the bound from part (b) over k)

   (d) [**Extra Credit**] Prove this for $p = (1+\epsilon)\ln n/n$.

4. **Degree-3 vertices**

   Consider a random graph from the distribution $G(n,p)$, with $p = 1/n$. We will show that it is very likely that such a graph contains a vertex of degree at least 3. (Actually, we could do this for much lower values of p)

   (a) Prove that there is a constant c such that the expected number of vertices of degree at least 3 is at least cn. (it suffices to do this for n large)

   Hint: Compute a lower bound on the probability that a vertex has degree at least 3. You could do this by computing the probability that a vertex has degree exactly 3, or by computing the probability that a vertex has degree 0, 1, or 2.

(b) Compute an upper bound on the variance of the number of vertices of degree at least 3. (it suffices to do this for n large)

(c) Use the variance bound from part (b) to prove that the probability there is vertex of degree at least 3 goes to 1 as n goes to infinity.

We will now see an alternate way to prove that the probability of a vertex of degree at least 3 goes to 1.

(d) Assume n is even and arbitrarily divide the vertices in half, into sets A and B where $|A| = n/2$. Find a constant c such that the probability that a vertex in A has at least 3 neighbors in B is at least c. Now, prove that the probability that there is no vertex in A with at least 3 neighbors in B goes to zero as n goes to infinity.

5. **Percolation on the Hexagonal Grid**

Consider percolation on the infinite hexagonal grid, of which a section is shown below.

(a) Prove that there is a constant $p_0 > 0$ such that if $p < p_0$ then the probability that the origin lies in an infinite component is zero.

(b) Prove that there are constants $p_1 < 1$ and $c > 0$ such that if $p > p_1$ then the probability that the origin lies in an infinite component is at least c.