Graphs and Networks **out:** November 9, 2006

Problem Set 4-Ugrad

Lecturer: Daniel A. Spielman **due:** November 28, 2006

1 Homework Policy

You may discuss the problems with other students. But, you must write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

You **may not** search the web for solutions to similar problems given out in other classes. If you think this policy needs any clarification, please let me know.

2 Corrections Made

1. (11/13/06) Clarified definition of $\mathrm{sp}(S)$ in problem 4.

3 Problems

1. **Sparsity and Conductance** For a graph $G = (V, E)$ and a partition of its vertices $C_1, \ldots, C_k$, we define the sparsity of the partition by

$$\mathrm{sp}(C_1, \ldots, C_k) = \sum_{i=1}^{k} \frac{|\partial(C_i)|}{|C_i|},$$

where $\partial(C_i)$ denotes the set of edges leaving C_i. Define the conductance of a partition by

$$\phi(C_1, \ldots, C_k) = \sum_{i=1}^{k} \frac{\mathrm{vol}(\partial(C_i))}{\mathrm{vol}(C_i)},$$

where vol() of a set of edges is the sum of their weights, and vol() of a set of vertices is the sum of their weighted degrees.

Describe a graph in which the partition C_1, C_2 minimizing $\mathrm{sp}(C_1, C_2)$ and the partition D_1, D_2 minimizing $\phi(D_1, D_2)$ satisfy $C_1 \cap D_1 = \emptyset$ and $|C_2 \cap D_2| > n/4$, where n is the number of vertices in the graph.

If you cannot prove that the obviously minimizing partitions are minimizing, just make your intuitive argument as precise as possible.

2. **Assortativity** Given a graph $G = (V, E)$, we define its assortativity as follows. Let p_k be the fraction of vertices of degree k. Let

$$q_k = \frac{kp_k}{\sum_j jp_j}.$$

The quantity q_k is the probability that a random endpoint of a randomly chosen edge has degree k. (If you look at Newman's paper, he defines this as q_{k+1}, but I think he's got it wrong. In particular, his equation (4) agrees with my interpretation) Define $e_{j,k}$ to be the probability that a randomly chosen edge has degree j at its first endpoint and degree k at the other. (that is, we choose a random edge and a random endpoint)

Define

$$\sigma_q^2 \stackrel{\text{def}}{=} \sum_k k^2 q_k - \left(\sum_k kq_k\right)^2.$$

Finally, define

$$r(G) = \frac{1}{\sigma_q^2}\sum_{j,k} jk(e_{j,k} - q_j q_k).$$

a. find a graph G with $r(G) = 1$.

b. find a graph G with $r(G) = -1$.

3. Let G be a random directed graph on n vertices in which each vertex has out-degree 1. That is, for every vertex i, we choose a j at random and add edge (i, j) to the graph. For a vertex v, let $R(v)$ denote the set of vertices reachable from v.

a. Prove that for a fixed vertex v, $\mathrm{P}\left[|R(v)| < \sqrt{n}/10\right] < 1/3$.

b. Prove that for a fixed vertex v, $\mathrm{P}\left[|R(v)| > 10\sqrt{n}\right] < 1/3$.

Note that these constants are pretty loose. Much tighter bounds are possible.

4. **You can't cluster a random graph.** Consider a random graph distributed according to $G(n, p)$, with $p = 10 \ln n/(n - 1)$. Prove that there exists an absolute constant $\alpha > 0$ such that

$$\mathrm{P}\left[\min_S \mathrm{sp}(S) < \alpha\right] < 1/3,$$

where

$$\mathrm{sp}(S) = \frac{|\partial(S)|}{\min(|S|, |V - S|)}.$$

Note that you can also prove this for $\phi(S)$, it's only slightly trickier.