1 Homework Policy

You may discuss the problems with other students. But, you must write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

You may not search the web for solutions to similar problems given out in other classes. If you think this policy needs any clarification, please let me know.

2 Corrections Made

1. (11/13/06) Clarified definition of sp(S) in problem 4.

3 Problems

1. Sparsity and Conductance For a graph \( G = (V, E) \) and a partition of its vertices \( C_1, \ldots, C_k \), we define the sparsity of the partition by

\[
\text{sp}(C_1, \ldots, C_k) = \sum_{i=1}^{k} \frac{|\partial(C_i)|}{|C_i|},
\]

where \( \partial(C_i) \) denotes the set of edges leaving \( C_i \). Define the conductance of a partition by

\[
\phi(C_1, \ldots, C_k) = \sum_{i=1}^{k} \frac{\text{vol}(\partial(C_i))}{\text{vol}(C_i)},
\]

where \( \text{vol()} \) of a set of edges is the sum of their weights, and \( \text{vol()} \) of a set of vertices is the sum of their weighted degrees.

Describe a graph in which the partition \( C_1, C_2 \) minimizing \( \text{sp}(C_1, C_2) \) and the partition \( D_1, D_2 \) minimizing \( \phi(D_1, D_2) \) satisfy \( C_1 \cap D_1 = \emptyset \) and \( |C_2 \cap D_2| > n/4 \), where \( n \) is the number of vertices in the graph.

If you cannot prove that the obviously minimizing partitions are minimizing, just make your intuitive argument as precise as possible.
2. **Assortativity** Given a graph $G = (V, E)$, we define its assortativity as follows. Let $p_k$ be the fraction of vertices of degree $k$. Let

$$q_k = \frac{k p_k}{\sum_j j p_j}.$$

The quantity $q_k$ is the probability that a random endpoint of a randomly chosen edge has degree $k$. (If you look at Newman’s paper, he defines this as $q_{k+1}$, but I think he’s got it wrong. In particular, his equation (4) agrees with my interpretation) Define $e_{j,k}$ to be the probability that a randomly chosen edge has degree $j$ at its first endpoint and degree $k$ at the other. (that is, we choose a random edge and a random endpoint)

Define

$$\sigma_q^2 \overset{\text{def}}{=} \sum_k k^2 q_k - \left( \sum_k k q_k \right)^2.$$

Finally, define

$$r(G) = \frac{1}{\sigma_q^2} \sum_{j,k} j k (e_{j,k} - q_j q_k).$$

a. find a graph $G$ with $r(G) = 1$.

b. find a graph $G$ with $r(G) = -1$.

3. Let $G$ be a random directed graph on $n$ vertices in which each vertex has out-degree 1. That is, for every vertex $i$, we choose a $j$ at random and add edge $(i, j)$ to the graph. For a vertex $v$, let $R(v)$ denote the set of vertices reachable from $v$.

a. Prove that for a fixed vertex $v$, $P \left[ |R(v)| < \sqrt{n}/10 \right] < 1/3$.

b. Prove that for a fixed vertex $v$, $P \left[ |R(v)| > 10\sqrt{n} \right] < 1/3$.

Note that these constants are pretty loose. Much tighter bounds are possible.

4. **You can’t cluster a random graph.** Consider a random graph distributed according to $G(n, p)$, with $p = 10 \ln n / (n - 1)$. Prove that there exists an absolute constant $\alpha > 0$ such that

$$P \left[ \min_S \text{sp}(S) < \alpha \right] < 1/3,$$

where

$$\text{sp}(S) = \frac{\left| \partial(S) \right|}{\min(|S|, |V - S|)}.$$

Note that you can also prove this for $\phi(S)$, it’s only slightly trickier.