Graphs and Networks **out:** November 29, 2006

Problem Set 5-Grad

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1 Corrections Made

1. None yet, but history suggest we will need this section.

2 Problems

1. Prove that the complete graph on 5 vertices is not planar.
2. Let $G = (V, E)$ be a 2-connected planar graph. Let $H = (F, E)$ be its dual (recall that the edges of H are in 1-1 correspondance with the edges of G). Let T be a spanning tree of G, and let S be the set of edges not in T. Prove that S is a spanning tree of H.
3. Let $\{v_1, \ldots, v_n\}$ be points in $\mathbb{R}^2$ such that no four lie on one circle. Let S be a sphere tangent to the plane $\mathbb{R}^2$ at its "south pole", and let p be its "north pole". Let Π be the stereographic projection map, which maps each point x in $\mathbb{R}^2$ to the intersection with S of the line through x and p. I claimed that this map sends circles in $\mathbb{R}^2$ to circles on S, and *vice versa*. You may assume this.

   In class, I claimed that the line segement from $\Pi(v_i)$ to $\Pi(v_j)$ is on the convex hull of the point set $\{p, \Pi(v_1), \ldots, \Pi(v_n)\}$ if and only if (v_i, v_j) is a Delaunay edge of $\{v_1, \ldots, v_n\}$.

   Prove it.
4. Let P be an infinite set of points in $\mathbb{R}^2$ such that

   a. for all $p, q \in P$, the distance between p and q is at least 1, and
   b. for every $x \in \mathbb{R}^2$, there is a $p \in P$ such that the distance from p to x is at most $\sqrt{2}$.

   Let T be the Delaunay triangulation of P. Prove that there is an $\alpha > 0$ such that the smallest angle of every triangle in T is at least α.
5. Prove that every planar graph can be drawn in the plane using non-intersecting straight line segments for every edge.

   Hint: Try induction on the number of vertices. When you remove a vertex, carefully choose some edges to add between its neighbors.

   Hint: Note that it suffices to consider planar graphs in which every face is a triangle.