In \( G(n, p) \), we expect \( \binom{n}{2} p \) edges,

so for \( p = \frac{1}{n} \), expect \( \frac{1}{2} \) edges --

so expected degree of each vertex is 1

Let's see that it is unlikely to deviate.

To bound chance it differs, apply Chernoff - Hoeffding bounds:

Let \( X_1, \ldots, X_k \) be \( k \) i.i.d. random variables

such that

\[
\Pr[X_i = 1] = p_i,
\]

\[
\Pr[X_i = 0] = 1 - p_i.
\]

Let \( X = \sum X_i \)

Let \( \mu = \sum p_i = E[X] \)

Then \( \forall \delta < 1 \)

\[
\Pr[X < (1 - \delta)\mu] < e^{-\frac{\delta^2 \mu}{2}}
\]

\[
\Pr[X > (1 + \delta)\mu] < e^{-\frac{\delta^2 \mu}{3}}
\]

Ex: create a variable for each potential edge

\( X = \text{total \# edges} \)

\( p = \frac{1}{n} \), \( \mu = \frac{n(n-1)}{2} \), \( p = \frac{n}{2} \)
So, $P(\text{edges} < \frac{n}{3}) = P(X < (1-\delta)n^2), \delta = \frac{1}{3}$

$$\leq e^{\frac{-\mu x^2}{2}} = e^{\frac{-n \cdot (1-\delta) n^2}{2} \cdot \frac{1}{2}} = e^{\frac{-n}{36}}$$

for $n$ big, is exceedingly small.

So, actual # edges tightly concentrated around $p(\frac{1}{2})$.

We will prove crude bounds on diameter of $G(n,p)$ for $p = 6 \ln n / n$

First, let's prove a lower bound on the diameter.
Let $d_{\text{max}}$ be the maximum degree.

Claim 1: $P(\text{diam} > 11 \ln n) \to 0$ as $n \to \infty$.

**pf:** For any particular node, $E[d] = 6 \ln n = \mu$

So applying Chernoff bound with $\delta = \frac{5}{6}$, we have

$$P(\text{diam} > 11 \ln n) = P(\text{diam} > (1+\delta)E[d]) \
\leq e^{\frac{-\delta \mu}{3}} \
= e^{\frac{-25 \cdot 6 \ln n}{36 \cdot 3}} \leq n^{-25/18}$$

So, $P(\text{diam} > 11 \ln n) \leq n^{-25/18}$

\[ \cdots \]
let $B_r(x) = \{ u : d(u, x) \leq r \}$, \quad $B_0(x) = \emptyset$

Now, $|B_r(x)| \leq d_{\text{max}} \cdot |B_{r-1}(x)| \leq d_{\text{max}}^{r-1}$

So, if $d_{\text{max}} \leq \frac{n}{2}$, then most nodes have distance greater than $r$ to $x$.

Similarly, if $d_{\text{max}} < n$ then $\text{diam}(G) \geq 2r$,

happens if $r \geq \frac{\log n}{\log d_{\text{max}}} = \frac{\log n}{\log (\log n)} = \frac{\log n}{\log (\log n) + \log \log n}$

We will try to prove a comparable upper bound on the diameter of $\frac{\log n}{\log \log n} + 1$.

Our approach will have two steps.

Let $S_r(x) = B_r(x) - B_{r-1}(x)$

1. We will show that for $l = \frac{\log n}{2 \log \log n}$

   with high probability $|S_l(x)| \geq \sqrt{n}$.

2. Then, for each pair $x, y$

   with high probability there is an edge between $S_l(x)$ and $S_l(y)$
   (or they overlap)

Let's do part 2 first:

Given that $|S_l(x)| \geq \sqrt{n}$ and $|S_l(y)| \geq \sqrt{n}$,

and that they don't overlap,

There are at least $n$ pairs $(u, v) \in S_l(x), u \in S_l(y)$
There are at least \( n \) pairs \((u,v) \in \mathcal{E}(x), u \in \mathcal{E}(y)\)

The probability that none appears in the graph is at most

\[
(1-p)^n = (1 - \frac{6\ln n}{n})^n = \left(e^{-\frac{6\ln n}{n}}\right)^n = e^{-\frac{6\ln n}{n}}. 
\]

So, the probability there exists a pair \((x,y)\) with \(d_{\text{dist}}(x,y) > 2l + 1 \leq \left(\frac{9}{12}\right)^{1/6} \leq \sqrt{\frac{n}{6}}\)

---

Let's return to part 1.

We will show that, with high probability,

\[
f(r) \leq \frac{\ln n}{2\ln n}, \quad |S_r(x)| \geq (\ln n) |S_{r-1}(x)| .
\]

which implies \(|S_r(x)| \geq (\ln n)^r = (\ln n)^{\frac{\ln n}{2\ln n}} = \sqrt{n}^{\frac{1}{2}}\)

To do this, we consider the probability a node in \(V - B_{r-1}(x)\) is in \(S_{r}(x)\). The probability it is not in \(S_r(x)\) is the probability it has no edge from \(S_{r-1}(x)\)

\[
(1-p)^{|S_{r-1}(x)|} \leq 1 - p |S_{r-1}(x)| + \left(\frac{|S_{r-1}(x)|}{2}\right)p^2 \quad (*)
\]

To estimate this, first calculate

Assuming \(\max \leq \sqrt{\ln n}, \quad |B_{r-1}(x)| \leq (\ln n)^{\frac{1}{2}} = \sqrt{n}. \quad n^{\frac{1}{2}} \)

\[
\leq \sqrt{n},
\]

for \(n\) sufficiently large

And, for \(n\) sufficiently large, we get

\[
(*) \leq 1 - \left(\frac{11}{12}\right)p |S_{r-1}(x)|
\]

\[
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\]
So, the prob a node in \( V-B_t-1(t) \) is in \( S_t(t) \) is at least

\[
\frac{11}{12} \Pr(S_{t-1}(t)).
\]

For \( n \) sufficiently large, \( |B_t(t)| \leq \frac{n}{11} \), so

\[
|V-B_{t-1}(t)| \geq \frac{10}{11} n,
\]

and so the expected number of nodes in \( S_t(t) \) is at least \( \frac{10}{12} \cdot P \cdot n \cdot |S_{t-1}(t)| = \frac{5}{11} n \).

So, applying a Chernoff bound with \( \mu = \frac{5}{11} n \cdot |S_{t-1}(t)| \),

\[
\delta = \frac{\mu}{5}, \quad w \quad \text{we find}
\]

\[
\Pr \left[ |S_{t}(t)| \leq \left( 1 - \frac{\mu}{w} \right) |S_{t-1}(t)| \right] \leq e^{-\frac{\mu \delta}{2}} \leq \left( \frac{e^{-\frac{\mu \delta}{2}}}{\frac{5}{11} n} \right)^{|S_{t-1}(t)|},
\]

\[
\leq \frac{-8/5}{n}.
\]

To finish, we now observe that the probability there is an \( x \) and \( t \leq l \) for which \( |S_t(t)| < \left( \frac{\mu}{w} \right) |S_{t-1}(t)| \) is at most \( n \cdot \frac{-8/5}{n} = \frac{-8}{5} \) for \( n \) sufficiently large.

So, with prob at least \( 1 \left( \frac{\mu}{w} + \frac{\mu}{n} + \frac{\mu}{n} \right) \) for \( n \) sufficiently large, the diameter is at most \( \frac{\mu}{w} + 1 \).

Let me clean up a little bit of that logic. Let \( A \) be the event

\[
d_{\max} \leq 11 \ln n.
\]

We showed \( \Pr(\overline{A}) \leq \frac{-7/18}{n}. \)
let \( C_{x,t} \) be the event \( |S_t(x)| < (\ln n)(S_{t-1}(A)) \)
and \( D_{x,t-1} \) be the event \( |B_{t-1}(A)| > (11 \ln n)^{-1} \)

We proved \( \Pr[C_{x,t} \mid D_{x,t-1}] < n^{-8/5} \).

And, \( D_{x,t-1} \Rightarrow A \),

We want an upper bound on
\[
\Pr \left[ \bigoplus_{x,t} C_{x,t} \right], \text{ so write}
\]
\[
\Pr \left[ \bigoplus_{x,t} C_{x,t} \right] = \Pr \left[ A \text{ and } \bigoplus_{x,t} C_{x,t} \right] +
\]
\[
\Pr \left[ A \text{ and } \bigoplus_{x,t} C_{x,t} \right]
\]
\[
= n^{-7/18} + \Pr \left[ \bigoplus_{x,t} (A \text{ and } C_{x,t}) \right]
\]

now, \( \Pr[\overline{A} \text{ and } C_{x,t}] \)

as \( A \Rightarrow D_{x,t-1} \)

\[
\leq \Pr \left[ D_{x,t-1} \text{ and } C_{x,t} \right]
\]
\[
= \Pr[C_{x,t} \mid D_{x,t-1}] \cdot \Pr[D_{x,t-1}]
\]
\[
\leq \Pr[C_{x,t} \mid D_{x,t-1}]
\]
So,

\[ \Pr \left[ \bigcup \limits_{x \in T} C_{x,r} \right] \leq n^{-8/5} \]

\[ n^{-7/8} + l \cdot n \cdot n^{-8/5} \]

\[ \leq n^{-7/8} + n^{-1/12} \text{ for } n \text{ sufficiently large.} \]