Physicists introduced percolation theory to help answer the following question: if a porous rock is immersed in water, will the water reached the center of the rock? To answer this question, they modeled the rock by a grid graph. We will model a two-dimensional rock by a two-dimensional grid graph. The internal structure of the rock is assumed to be random. Water can potentially flow along the edges. Each edge is chosen to be "open" with probability $p$ and "closed" with probability $1-p$. Water can flow along the open edges but not along closed edges.

As the boundary of the graph is in contact with water, water will reach the center of the rock if the center vertex is connected by a path of open edges to the boundary.

Instead of considering finite grid graphs, physicists prefer to consider the infinite grid. They then ask how large $p$ needs to be before some given vertex is probably contained and in an infinite component of open edges. They called the critical probability the probability below which this is zero and above which it is finite. For the two-dimensional grid, the critical probability turns out to be one half.

In today's lecture we will consider percolation on infinite trees. We will start with the infinite binary tree. I will present three proofs that the critical probability for the infinite binary tree is one half.

Percolation on trees is related to many other interesting things, including:
- Population dynamics
- The spread of epidemics
- The formation of the giant component in random graphs.

In the next lecture, we will use our study of percolation on trees to prove that

$$\text{for } t > 0, \exists \varepsilon > 0, \text{ s.t. } \forall p = \frac{1+t\varepsilon}{n},$$

$$\Pr \left[ \text{largest component of } G \text{ has } \geq \text{cn vertices} \right] \rightarrow 1$$

and for $p = \frac{1-\varepsilon}{n}$

$$\Pr \left[ \text{largest component of } G \text{ has } \leq c_0 \sqrt{n} \text{ vertices} \right] \rightarrow 1$$
Instead of thinking about edges being opened or closed, as the physicists do, I will talk about edges being present or not present. To be formal I will identify the vertices of the complete binary tree with the strings over the alphabet 0/1.

\[
V = \mathcal{X}_{0,1}^{\mathbb{N}} = \bigcup_{n \geq 0} \mathcal{X}_{0,1}^{n}
\]

\((x,y) \in E \text{ if } y = x0 \text{ or } x1\)

\[\phi_0 \circ \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n \circ \cdots\]

etc.

Let \(C\) be the component of \(\phi_i\), and let

\[\Theta(p) = \mathbb{P} \left[ \frac{|C| = \infty}{P} \right]\]

\[P_c = \sup \{p : \Theta(p) > \frac{3}{2}\}\]

We will now show that \(P_c = \frac{1}{2}\).

In particular, we will show that

\[\Theta(p) = \begin{cases} 0 & p \leq \frac{1}{2} \\ \frac{2p-1}{p^2} & p > \frac{1}{2} \end{cases}\]
Let $A_0$ be the event that node 0 is in an infinite component "going forward" that is, not involving $\phi$. Let $A_1$ be the same for 1.

By self-similarity

$$P(\overline{A_0}) = P(\overline{A_1}) = \Theta(p)$$

Let $B_i$ be the event that there is an edge from $\phi$ to $i$ and that $A_i$ holds. Then

$$P(\overline{B_i}) = p\Theta(p).$$

As $B_0$ and $B_1$ are independent,

$$\Theta(p) = P(\overline{B_0} \lor B_1)$$
$$= P(\overline{B_0} \overline{B_1}) + P(B_0 \overline{B_1}) - P(\overline{B_0} B_1)$$
$$= 2p\Theta(p) - (p\Theta(p))^2.$$ 

One solution of this equation is $\Theta(p) = 0$. If $\Theta(p) = 0$, we can divide by $\Theta(p)$ to get
In case you don't, or in case you are uncomfortable and infinite trees, we will now do a more concrete proof. We will consider the infinite tree as the limit of finite trees of increasing depth. Let

\[ T^0_2 \] be the tree of depth 0,

so \[ T^0_2 = \emptyset, \quad T^1_2 = \emptyset, \quad \text{etc.} \]

Let \( \Theta_d(p) = \mathbb{R}^\infty \{ \text{exists a path from } \emptyset \text{ to a leaf of } T^d_2 \} \)

We can either prove or take as a definition

\[ \Theta(p) = \lim_{d \to \infty} \Theta_d(p). \]

Let's first prove, for \( p < \frac{1}{2} \)

\[ \lim_{d \to \infty} \Theta_d(p) = 0 \]
We have

\[ \Pr[\text{exists path from } 0 \text{ to a leaf}] \]

\[ \leq \sum_{x \in \mathbb{N}^{d}} \Pr[\text{each edge on path } 0 \to x \text{ appears}] \]

\[ = \sum_{x \in \mathbb{N}^{d}} p^d = 2^d p^d = (2p)^d \]

\[ \to 0 \quad \text{if} \quad p < \frac{1}{2}. \]

Now, I'll show you that for \( p > \frac{1}{2} \)

\[ \lim_{d \to \infty} \Theta_d(p) = \frac{2p-1}{p^2} \]

Let's compute the first few values. By the previous analysis, we have

\[ \Theta_1(p) = 2p - \Theta_0(p) - p^2 \Theta_2(p)^2 \]

or \[ \Theta_d(p) = f(\Theta_{d-1}(p)), \] where

\[ f(x) = 2px - p^2x^2 \]

\[ \Theta_0(p) = 1 \]

\[ \Theta_1(p) = 2p - p^2 \]
Let \( x^* = \frac{2p-1}{p^2} \). We know that

1. \( f(x^*) = x^* \), and
2. \( \Theta_0(p) = 1 > x^* \)

To show that \( \Theta_d(p) > x^* \) for all \( d \), I will show

\[
f(x) > x^* \quad \text{for all} \quad x > x^*
\]

As \( f(x^*) = x^* \), it suffices to show that \( f \) is increasing on \([x^*, 1]\).

So, look at the derivative:

\[
f'(x) = 2p - 2px = 2p(1-px) > 0
\]

As \( p, x < 1 \).

So, \( f'(x) \) is increasing for \( x < 1 \), which suffices

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In the rest of the lecture, we will consider \( k \)-ary trees.

These have critical probability \( p_c = \frac{1}{k} \).
These have critical probability $p_c = \frac{1}{2}$.

For $p < p_c$, we will examine the size of the component $G$.

To do this, imagine a process in which each vertex in the component of $G$ goes through two states. It begins asleep, at some time it wakes up, has some children, and then retires.

Each node has $k$ opportunities to have children, and succeeds in each opportunity with probability $p$.

We will assign a number to each vertex according to when it appears. We assign number 1 to node 0. If it has $j$ children, we assign them numbers $1, \ldots, j$. Each time a node has children, we assign its children the next available numbers.

Let $y_i$ be the number of children of node $i$. So, the children of node $i$ begin at number $1 + y_1 + y_2 + \cdots + y_{i-1}$.

We consider a process in which exactly 1 node is active in each time step.
At step $t$, node $t$ has children and then retires.

The process dies out if at time $t$ there is no node numbered $t$. Let $Z_t$ be the number of nodes present at the end of time $t$.

So, $Z_0 = 1$, $Z_t = 1 + Y_t$, $Z_\infty = 1 + Y_t + \cdots + Y_\infty$

The process dies if $Z_t < t + 1$ for some $t$. Let's consider the chance of this. That is

$$\Pr[Z_t < t + 1]$$

$$= \Pr[Y_t + \cdots + Y_\infty < t]$$

We have $Y_i = X_{i,1} + \cdots + X_{i,k}$

where $X_{i,j} \sim \text{i.i.d. Bern}(p)$ with $p \leq 0.6$.

So, $\mu = E[Y_t + \cdots + Y_\infty] = \frac{pt}{k} \times t$.

If we set $p = \frac{(\alpha e)}{k}$, then
\[ \Pr \left[ Y_1 + \cdots + Y_t < t \right] = 1 - \Pr \left[ Y_1 + \cdots + Y_t \geq t \right], \]

and

\[ \Pr \left[ Y_1 + \cdots + Y_t \geq t \right] = \Pr \left[ Y_1 + \cdots + Y_t \geq \mu \left( 1 - \frac{1}{t} \right) \right] \]

\[ = \Pr \left[ Y_1 + \cdots + Y_t \geq (1 + \delta) \mu \right], \text{ where} \]

\[ \delta = \frac{1}{1 - \varepsilon} - 1 \geq \varepsilon. \]

Applying the Chernoff bound, we find that

probability is at most

\[ \frac{-\varepsilon^2 \mu}{e^{\frac{t}{3}}} = e^{-\varepsilon^2 (1 - \varepsilon) t / 3}, \]

which becomes very small as \( t \) grows.

So, it is exponentially unlikely that the component becomes big.