Lecture 14: percolation on trees

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Physicists introduced percolation theory to help answer the following question: if a porous rock is immersed in water, will the water reached the center of the rock? To answer this question, they modeled the rock by a grid graph. We will model a two-dimensional rock by a two-dimensional grid graph. The internal structure of the rock is assumed to be random. Water can potentially flow along the edges. Each edge is chosen to be "open" with probability p and "closed" with probability 1-p.water can flow along the open edges but not along closed edges.

As the boundary of the graph is in contact with water, water will reach the center of the rock if the center vertex is connected by a path of open edges to the boundary.

Instead of considering finite grid graphs, physicists prefer to consider the infinite grid. They then ask how large p needs to be before some given vertex is probably contained and in an infinite component of open edges. They called the critical probability the probability below which this is zero and above which it is finite. For the two-dimensional grid, the critical probability turns out to be one half.

In today's lecture we will consider percolation on infinite trees. We will start with the infinite binary tree. I will present three proofs that the critical probability for the infinite binary tree is one half.

Percolation on trees is related to many other interesting things, including:

- Population dynamics
- The spread of epidemics
- The formation of the giant component in random graphs.

In the next lecture, we will use our study of percolation on trees to prove that

for $\forall \varepsilon > 0$, $\exists$ c s.t. for $p = \frac{1+\varepsilon}{n}$

$$\Pr_{G \leftarrow G(n,p)} [\text{largest component of } G \text{ has} \geq cn \text{ vertices}] \to 1$$

and for $p = \frac{1-\varepsilon}{n}$

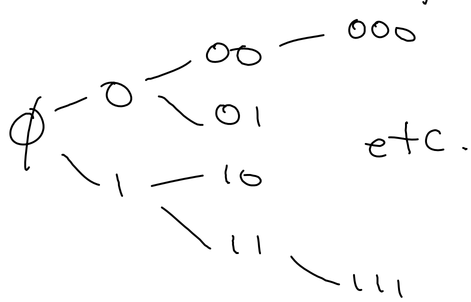
$$\Pr [\text{largest component of } G \text{ has} \leq c \log_2 n \text{ vertices}] \to 1$$

$G \leftarrow G(n,p)$

Instead of thinking about edges being opened or closed, as the physicists do, I will talk about edges being present or not present. To be formal I will identify the vertices of the complete binary tree with the strings over the alphabet 0/1.

$$V = \{0,1\}^* = \bigcup_{n \geq 0} \{0,1\}^n$$

$(x,y) \in E$ if $y = x0$ or $x1$



Let C be the component of ϕ, and let

$$\theta(p) = \Pr_p[\,|C| = \infty\,]$$

$$p_c = \sup\{p : \theta(p) = 0\}$$

We will now show that $p_c = \frac{1}{2}$. In particular, we will show that

$$\theta(p) = \begin{cases} 0 & p \leq \frac{1}{2} \\ \frac{2p-1}{p^2} & p > \frac{1}{2} \end{cases}$$

Let A_0 be the event that node 0 is in an infinite component "going forward" – that is, not involving ϕ.
Let A_1 be the same for 1.

By self-similarity

$$\Pr[A_0] = \Pr[A_1] = \theta(p)$$

Let B_i be the event that there is an edge from ϕ to i and that A_i holds. Then

$$\Pr[B_i] = p\theta(p).$$

As B_0 and B_1 are independent,

$$\begin{aligned}\theta(p) &= \Pr[B_0 \vee B_1] \\ &= \Pr[B_0] + \Pr[B_1] - \Pr[B_0 \wedge B_1] \\ &= 2p\theta(p) - (p\theta(p))^2.\end{aligned}$$

One solution of this equation is $\theta(p) = 0$.
If $\theta(p) \neq 0$, we can divide by $\theta(p)$ to get

$$1 = 2p - p^2\theta(p), \quad \text{or}$$

$$\theta(p) = \frac{2p-1}{p^2}.$$

So, for $p > \frac{1}{2}$, $\frac{2p-1}{p^2}$ is a solution.

But, do we believe this implies $\theta(p) = \frac{2p-1}{p^2}$?

In case you don't, or in case you are uncomfortable and infinite trees,we will now do a more concrete proof. We will consider the infinite tree as the limit of finite trees of increasing depth. Let

T_2^d be the tree of depth d,

so $T_2^0 = \phi$, $T_2^1 = \phi <^{0}_{1}$, etc.

Let $\theta_d(p) = \Pr_p[\exists$ a path from ϕ to a leaf of $T_2^d]$

We can either prove or take as a definition

$$\theta(p) = \lim_{d\to\infty} \theta_d(p).$$

Let's first prove, for $p < \frac{1}{2}$

$$\lim_{d\to\infty} \theta_d(p) = 0$$

We have

$$\Pr[\exists \text{ a path from } 0 \text{ to a leaf}]$$

$$\leq \sum_{x \in \{0,1\}^d} \Pr[\text{each edge on path } 0 \text{ to } x \text{ appears}]$$

$$= \sum_{x \in \{0,1\}^d} p^d = 2^d p^d = (2p)^d$$

$$\to 0 \quad \text{if} \quad p < \tfrac{1}{2}.$$

Now, I'll show you that for $p \geq \frac{1}{2}$

$$\lim_{d \to \infty} \Theta_d(p) \geq \frac{2p-1}{p^2}$$

Let's compute the first few values. By the previous analysis, we have

$$\Theta_d(p) = 2p\,\Theta_{d-1}(p) - p^2\,\Theta_{d-1}(p)^2$$

or $\Theta_d(p) = f(\Theta_{d-1}(p))$, where

$$f(x) = 2px - p^2x^2$$

$$\Theta_0(p) = 1$$

$$\Theta_1(p) = 2p - p^2$$

etc.

Let $x^* = \frac{2p-1}{p^2}$. We know that

1. $f(x^*) = x^*$, and
2. $\theta_0(p) = 1 > x^*$

To show that $\theta_d(p) > x^*$ for all d, I will show

$$f(x) > x^* \text{ for all } x > x^*$$

As $f(x^*) = x^*$, it suffices to show that f is increasing on $[x^*, 1]$.

So, look at the derivative:

$$f'(x) = 2p - 2p^2 x = 2p(1-px) > 0$$

as $p, x < 1$.

So, $f'(x)$ is increasing for $x < 1$, which suffices

In the rest of the lecture, we will consider k-ary trees.

These have critical probability $p_c = \frac{1}{k}$.

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For $p < p_c$, we will ~~examine~~ the size of the ~~component of~~ ϕ.

To do this, imagine a process in which each vertex in the component of ϕ

Goes through two states. It begins asleep, at some time it wakes up, has some children, and then retires.

Each node has k opportunities to have children, and succedes in each opportunity with probability p.

We will assign a number to each vertex according to when it appears. We assign number 1 to node ϕ. If it has j children, we assign them numbers $1, .., j$. Each time a node has children, we assign its children the next available numbers.

Let Y_i be the number of children of node i. So, the children of node i begin at number
$1 + Y_1 + Y_2 + \cdots + Y_{i-1}$.

We consider a process in which exactly 1 node is active in each time step.

At step t, node t has children and then retires.

The process dies ~~out~~ if at time t there is no node numbered t. Let Z_t be the number of nodes present at the end of time t.

So, $Z_0 = 1$, $Z_1 = 1 + Y_1$, $Z_t = 1 + Y_1 + \cdots + Y_t$

The process dies if $Z_t < t+1$ for some t. Let's consider the chance of this. That is

$$\Pr[Z_t < t+1]$$

$$= \Pr[Y_1 + \cdots + Y_t < t]$$

We have $Y_i = X_{i,1} + \cdots + X_{i,k}$

where $X_{i,j} = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{o.w.} \end{cases}$

So, $\mu = E[Y_1 + \cdots + Y_t] = pkt$.

If we set $p = \frac{(1-\varepsilon)}{k}$, then

$$\Pr[Y_1 + \cdots + Y_t < t] = 1 - \Pr[Y_1 + \cdots + Y_t \geq t],$$

and

$$\Pr[Y_1 + \cdots + Y_t \geq t] = \Pr\left[Y_1 + \cdots + Y_t \geq \mu\left(\frac{1}{1-\varepsilon}\right)\right]$$

$$= \Pr[Y_1 + \cdots + Y_t \geq (1+\delta)\mu], \text{ where}$$

$$\delta = \frac{1}{1-\varepsilon} - 1 \geq \varepsilon.$$

Applying the Chernoff bound, we find this probability is at most

$$e^{-\frac{\varepsilon^2 \mu}{3}} = e^{-\varepsilon^2 (1-\varepsilon) t / 3},$$

which becomes very small as t grows.

So, it is exponentially unlikely that the component becomes big.