Percolation in the Grid

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11:07 AM

The critical probability for percolation in the two-dimensional grid is one-- half. Today, I will prove to you that it is between one third and two thirds. I will then explain how one can show that it is at least one half.I will have to leave some details out of this last argument.

Fix the origin of the two-dimensional grid. We want to show that when the probability of an edge is less than one third, the probability that the origin is in an infinite component is zero. In particular, we will prove that the probability that there is an infinite simple path from the origin is zero. If there were an infinite path containing the origin, then there would be a path of every finite length containing the origin. we will bound this probability for some particular large length.

Let P be a simple path of length n. The probability that every edge of P appears in the graph is $(p)^n$. We will now bound the expected number of simple paths of length n that start at the origin and appear in the graph.

To do this, we will count the number of simple paths of length n that started the origin. Actually, we will just prove an upper bound on the number of such paths. Given that the path starts at the origin, there are four choices for its first edge. As the path cannot double back upon itself, there are three choices for the next edge, and three choices for every successive edge. So, the number of such paths is at most

$$4 \cdot 3^{n-1}$$

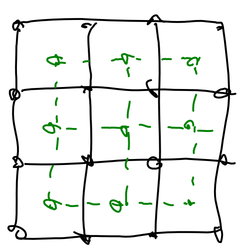
So, the expected number of these that appear in the graph is at most

$$4 \cdot 3^{n-1} p^n = 4p(3p)^{n-1} \xrightarrow[n \to \infty]{} 0 \quad \text{for } p < \tfrac{1}{3}.$$

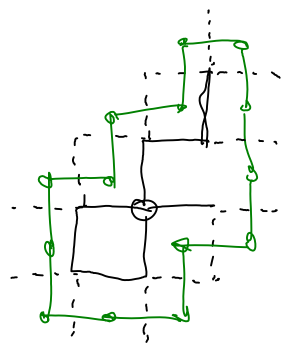
As the probability that the origin is in a path of length n is at most the expected number of such paths, the probability that the origin is in a path of length n goes to zero as n goes to infinity.

I now want to convince you that the critical probability is less than or equal to two thirds. That is, I will show you that if p is greater than two thirds then there is some nonzero probability that the origin is in an infinite component.

This argument will involve considering the dual of the grid. The dual of the grid graph is another grid graph. It has a vertex at the center of every square of the original grid, and an edge crossing every edge of the original grid. For example, here is a small grid and its dual.



Of course, the dual of the infinite grid is also an infinite grid. We say that a dual edge is dual to the primal edge that it crosses. When considering percolation problems, we include an edge in the dual graph if and only if we do not include it in the primal. The key fact that we will use about the dual is that if one considers the component of the origin, if the dual edges to its boundary form a cycle around the component. For example:



So, the origin is in an infinite component if and only if there is no cycle in the dual graph containing the origin. We will prove that when p is greater than two thirds, there is a nonzero probability that the dual does not contain a cycle containing the origin. We will first show that it is unlikely that the dual contains a large cycle around the origin.

Let $q = 1-p$, so each edge is in the dual with probability $q < \frac{1}{3}$.

Let C be any simple cycle of length n that contains the origin. The probability that C is in the graph is

$$q^n.$$

Now, let's bound the expected number of such cycles. Let S_n be the set of all simple cycles in the dual grid that contain the origin and have n edges. We need to upper bound $|C|$.

To do this, note that each such cycle must contain a dual vertex of form

$\left(\frac{1}{2}, \frac{1}{2}+j\right) \quad 0 \le j \le n-1.$

For each cycle, we consider its start vertex to be the vertex of largest j of this form that it contains. After the start vertex, there are 3 choices for the next edge. There are then at most 3 choices for each successive edge, so

$$|S_n| \le n \cdot 3^{n-1}$$

So, the expected number of simple dual cycles containing the origin of length n is

$$n \cdot 3^{n-1} \cdot q^n,$$

and the expected number of simple dual cycles containing the origin of length $\ge n$ is at most

$$\frac{1}{3} \sum_{j=n}^{\infty} j (3q)^j \longrightarrow 0 \quad \text{as } n \to \infty$$

To see this, set $x = \sum_{j=n}^{\infty} j (3q)^j$

Then $x = \frac{(3q)^n}{1-3q}\left(\frac{1}{1-3q} + (n-1)\right) \longrightarrow 0$

for $q < \frac{1}{3}$, $n \to \infty$

We are almost done. We can also show

that there is some non-zero chance that there are no short cycles. But, we need to combine these probabilities.

Here is a hack that makes it work.
Let B_n be the box of points $[-n,n]\times[-n,n]$.
Let
Y_n be the event that there is no simple dual cycle that encloses this box.
Our preceeding analysis tells us
$$\Pr[Y_n] \to 1 \text{ as } n \to \infty$$

In particular, $Y_n > 0$ for some n.

Let X_n be the event that all primal edges in this box appear. As there are finitely many such edges

$$\Pr[X_n] > 0.$$

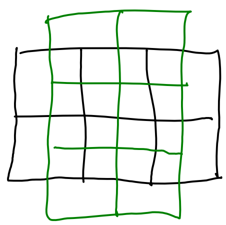
Now, if X_n and Y_n hold, then there is no dual cycle containing the origin. Moreover, these events depends on disjoint sets of edges, and so are independent.
So,
$$\Pr[0 \text{ in inf comp}] \geq \Pr[X_n \text{ and } Y_n]$$
$$= \Pr[X_n]\Pr[Y_n] > 0.$$

Note that we apply this argument for some finite n depending on q.

I will now explain how one proves that the critical probability is at least one half. We will prove that when p equals one half, there probably is a dual cycle containing the origin. the first step in our proof, is to observe something interesting that happens when p equals one-half. Consider an n -- by -- n grid. We will show that the chance that it contains a path from the left side to the right side is at least one half. To do this, we actually consider an

$n \times (n+1)$ grid,

and it's dual $(n+1) \times n$ grid:



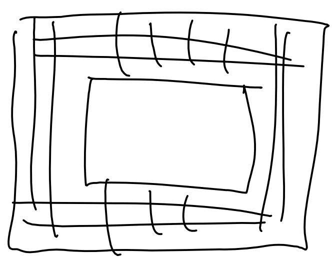
Notes that there is a path from the left to the right in the primal grid if and only if there is no path from the top to the bottom in a dual grid.

When $p = \frac{1}{2}$, there is symmetry between the two grids, so the probability of a LR path = the probability of an UD path, so both $= \frac{1}{2}$.

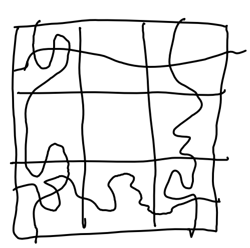
The chance of a LR path in an n-by-n grid is only slightly larger.

Using this fact, and some fancy probability, one can show that the probability of a LR path is an $n \times 3n$ grid is at least some constant, independent of n. Call it τ

So, consider the dual graph in $B_{3n} - B_n$



If there are LR paths in the top and bottom, and UD paths on the left and right, then the origin is contained in a simple cycle



What is the chance all 4 of these paths appear? We would like to say that it is at least τ^4.
But, we cannot immediately say this because they are not independent.

To say this, we need to know that these events are positively correlated

To prove this, we use the famous ~~FHG~~ inequality. It says

"If A and B are monotone graph properties, then

$$\Pr[A \text{ and } B] \geq \Pr[A]\Pr[B]$$"

Where a monotone graph property is one that only becomes truer when you add edges.

So, the probability of this ring $\geq \tau^4$.

We now consider the rings in

$$B_{3^k} - B_{3^{k-1}}$$

as these are disjoint, the probability that there is no cycle in the first k is at most

$$(1-\tau^4)^k \xrightarrow[k\to\infty]{} 0$$