Flocking

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A vector v_t evolving over time.

Have a graph G_t controlling evolution at time t.

Evolution rule is

$$v_{t+1}(i) = \frac{1}{d_t(i)+1}\left(v_t(i) + \sum_{(i,j)\in G_t} v_t(j)\right)$$

where $d_t(i)$ is deg of i in graph G_t.

In matrix form,

$$v_{t+1} = M_t v_t, \text{ where}$$

$$M_t = (D_t + I)^{-1}(A_t + I).$$

Note that this is the transpose of a walk-matrix.

<u>Theorem:</u>

If all graphs G_t connected, all nodes converge to same value.

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In particular,

We will prove that for $t = k(n-1)$, $\forall\, i, j$

$$|v_t(i) - v_t(j)| \le \|v_0\|_\infty \left(1 - \frac{1}{n^{n-2}}\right)^k$$

$$\le \|v_0\|_\infty \exp\left(- \frac{k}{n^{n-2}}\right).$$

Recall that $\|v_0\|_\infty = \max_i |v_0(i)|$

Let $P_t = M_t \cdot M_{t-1} \cdot \cdots \cdot M_1$

Let e_i denote elem unit vector.
Note $v_t(i) = e_i^T P^t v_0$, so

$$v_t(i) - v_t(j) = (e_i - e_j)^T P^t v_0.$$

We will show $\|(e_i - e_j)^T P^t\|_1 \le 2\left(1 - \frac{1}{n^{n-2}}\right)^k$,

which implies the result as
$$x^T y \le \|x\|_1 \cdot \|y\|_\infty$$

$\left(\text{Recall } \|x\|_1 = \sum_i |x(i)|\right)$

It will be easier to think about

$e_i P^t$, as it is related to random walks (but the graph changes at every step)

Here is the list of Lemmas we will use in the proof:

Lem 1 For all i and j,

$$e_i^T P_{n-1} e_j \geq \left(\frac{1}{n}\right)^{n-1}.$$

Lem 2 For all i and j,

$$\| (e_i - e_j) P_{n-1} \|_1 \leq 2\left(1 - \frac{1}{n^{n-2}}\right)$$

Lem 3 For all x s.t. $x^T \mathbb{1} = 0$,

$$\| x P_{n-1} \|_1 \leq \left(1 - n^{-(n-2)}\right) \|x\|_1$$

Lem 4 For all $k \geq 1$ and x s.t. $x^T \mathbb{1} = 0$,

$$\| x P_{k(n-1)} \|_1 \leq \left(1 - n^{-(n-2)}\right)^k \|x_1\|$$

To see why Lem 4 implies the theorem, let $x = e_i - e_j$, and note

$x^T \mathbb{1} = 0$ and $\|x\|_1 = 2$.

Proof of lem 1

First, note that every non-zero entry of M_i is at least $1/n$. Moreover, the diagonals of each M_i are all non-zero. We need to establish that each entry of P_{n-1} is $\geq \frac{1}{n^{n-1}}$.

To do this, define a $G_1, \ldots G_t$ path to be a sequence of vertices
$i_0, i_1, \ldots, i_t$ such that for each $1 \leq s \leq t$
either
(i_{s-1}, i_s) is an edge in G_s, or
$i_{s-1} = i_s$

If there is a $G_1 \ldots G_t$ path from i to j, then $P_t(i,j) > 0 \Rightarrow P_t(i,j) \geq n^{-t}$.

So, we just need to show that for each i, there is a $G_1, \ldots G_{n-1}$ path to every j.
We may prove this by induction.
Let $A_{i,s}$ be the set of vertices reachable from i by a $G_1, \ldots G_s$ path. As G_{s+1} is connected, we have $|A_{i,s+1}| \geq |A_{i,s}| + 1$.

Proof of lemma 2

First, note that $e_i^T P_t \geq 0$ (in each coord),
and $e_i^T P_t \mathbb{1} = 1$, for all t.
(as $P_t \mathbb{1} = \mathbb{1}$).

Next, note that for $x \geq 0 \geq y$,

$$|x+y| = |x| + |y| - 2\min(|x|,|y|).$$

So,

$$\|(e_i - e_j)^T P_{n-1}\|_1 = \|e_i^T P_{n-1} - e_j^T P_{n-1}\|_1$$

$$= \sum_k |e_i^T P_{n-1} e_k - e_j^T P_{n-1} e_k|$$

$$= \sum_k e_i^T P_{n-1} e_k + \sum_k e_j^T P_{n-1} e_k \qquad (*)$$

$$- 2 \sum_k \min(e_j^T P_{n-1} e_k, e_i^T P_{n-1} e_k) \qquad (**)$$

Now, $(*) = e_i^T P_{n-1} \mathbb{1} + e_j^T P_{n-1} \mathbb{1} = 2$,

and $(**) \leq -2n(n^{-(n-1)})$ by lem 1,

$$\leq -2n^{-(n-2)}$$

Proof of lem 3

pf. by induction on number of non-zero elements of x. Base has 2, which we already did.

(among non-zeros)

Let i minimize $|x(i)|$, and assume wolog $x(i) < 0$.

Consider any j s.t. $x(j) \geq 0$.

We have $x(j) > -x(i)$.

Set $y(k) = \begin{cases} x(i) & \text{if } i = k \\ -x(i) & \text{if } i = j \\ 0 & \text{o.w} \end{cases}$

and set $z = x - y$.

y has 2 non-zeros, and z has one less than x. Also

$$\|x\|_1 = \|z\|_1 + \|y\|_1 .$$

So

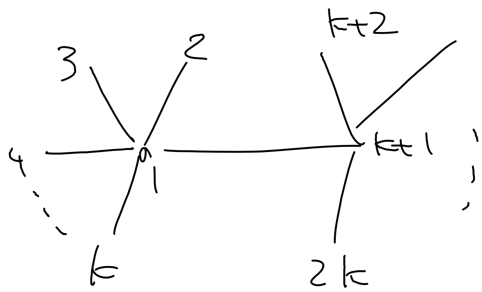
$$\|xP^{n-1}\|_1 \leq \|yP^{n-1}\|_1 + \|zP^{n-1}\|_1$$

$$\leq \gamma \|y\|_1 + \gamma \|z\|_1 = \gamma \|x\|_1$$

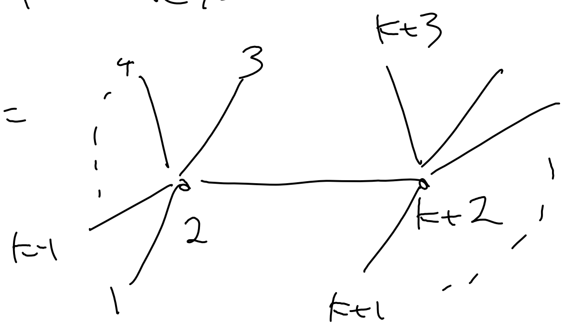
where $\gamma = 1 - \frac{1}{n^{n-2}}$

Lemma 4 follows from lemma 3 by induction on k

Example that does not mix at time $2^{n/2}$:

$G_1 =$ 

Then, rotate, so

$G_2 =$ 

and so on...