A vector $u_t$ evolving over time,

have a graph $G_t$ controlling evolution at time $t$,

Evolution rule is

$$u_{t+1}(i) = \frac{1}{d_t(i)+1} \left( u_t(i) + \sum_{(i,j)\in G_t} u_t(j) \right)$$

where $d_t(i)$ is degree of $i$ in graph $G_t$.

In matrix form,

$$u_{t+1} = M_t u_t,$$

where

$$M_t = (D_t + I)^{-1} (A_t + I).$$

Note that this is the transpose of a walk matrix.

Theorem: If all $G_t$ connected, all nodes converge to same value.
In particular, we will prove that for $t = k(n^2)$, i.e.,

$$|v_{t+1}(i) - v_{t+1}(j)| \leq \|\text{Vol}_{\phi}\| \left(1 - \frac{1}{n^2}\right)^k$$

$$\leq \|\text{Vol}_{\phi}\| \exp \left(-\frac{k}{n^2}\right).$$

Recall that $\|\text{Vol}_{\phi}\| = \max_i |v_0(i)|$

Let $P_t = M_t \cdot M_{t-1} \cdots M_1$

Let $e_i$ denote elem unit vector.

Note $v_{t+1}(i) = e_i^T P_t v_0$, so

$$v_{t+1}(i) - v_{t+1}(j) = (e_i - e_j)^T P_t v_0.$$

We will show $\| (e_i - e_j)^T P_t \|_1 \leq 2 \left(1 - \frac{1}{n^2}\right)^k$,

which implies the result as

$$x^Ty \leq \|x\|_1 \cdot \|y\|_{\infty}$$

(Recall $\|x\|_1 = \sum \|x(i)\|$)
It will be easier to think about $e_i P^t$ as it is related to random walks (but the graph changes at every step).

Here is the list of lemmas we will use in the proof:

**Lemma 1.** For all $i$ and $j$,

$$e_i P_{n-1} e_j = \left(\frac{1}{n}\right)^{n-1}.$$

**Lemma 2.** For all $i$ and $j$,

$$\| (e_i - e_j) P_{n-1} \|_1 \leq 2 \left(1 - \frac{1}{n^{n-2}}\right).$$

**Lemma 3.** For all $x \neq 0$, $x^T I = 0$,

$$\| x P_{n-1} \|_1 \leq \left(1 - \frac{1}{n^{n-2}}\right) \| x \|_1.$$

**Lemma 4.** For all $k \geq 1$ and $x \neq 0$, $x^T I = 0$,

$$\| x P_{k(n-1)} \|_1 \leq \left(1 - \frac{1}{n^{n-2}}\right)^k \| x \|_1.$$

To see why Lemma 4 implies the theorem, let $x = e_i - e_j$, and note
\[ x^T 1 = 0 \quad \text{and} \quad ||x||_1 = 2. \]

**Proof of Lemma**

First, note that every non-zero entry of \( M_i \) is at least \( y_i \). Moreover, the diagonals of each \( M_i \) are all non-zero. We need to establish that each entry of \( P_{n-1} \) is \( \geq \frac{1}{n^{n-1}} \).

To do this, define a \( G_{i_1} \ldots G_{i_t} \) path to be a sequence of vertices \( i_0, i_1, \ldots, i_t \) such that for each \( i_s \), either:

- \((i_{s-1}, i_s)\) is an edge in \( G_s \), or
- \( i_{s-1} = i_s \)

If there is a \( G_{i_1} \ldots G_{i_t} \) path from \( i \) to \( j \), then \( P_{i,j} (i,j) > 0 \Rightarrow P_{i,j} (i,j) > n^{-t} \).

So, we just need to show that for each \( i \), there is a \( G_{i_1} \ldots G_{i_{n-1}} \) path to every \( j \). We may prove this by induction. Let \( A_{i,0} \) be the set of vertices reachable from \( i \) by a \( G_{i_1} \ldots G_{i_{n-1}} \) path. As \( G_{i_{n-1}} \) is connected, we have \( |A_{i,0}| = |A_{i,n-1}| \).
Proof of Lemma 2

First, note that $e_i^T \pi_t = 0$ (in each coord), and $e_i^T \pi_t 1 = 1$, for all $t$.
(as $\pi_t 1 = 1$).

Next, note that for $x \geq y$,
$$|x+y| = |x| + |y| - 2 \min(x, y).$$

So, $\| (e_i - e_j)^T \pi_{n-1} \|_1 = \| e_i^T \pi_{n-1} - e_j^T \pi_{n-1} \|_1$

$$= \sum_k |e_i^T \pi_{n-1} e_k - e_j^T \pi_{n-1} e_k|$$

$$= \sum_k e_i^T \pi_{n-1} e_k + \sum_k e_j^T \pi_{n-1} e_k$$

$$- 2 \sum_k \min(e_j^T \pi_{n-1} e_k, e_i^T \pi_{n-1} e_k)$$

Now, $$(*) = e_i^T \pi_{n-1} 1 + e_j^T \pi_{n-1} 1 = 2,$$

and $$(** \leq -2 n (n^{-cn^{-2}}) \text{ by lem 1},$$

$$\leq -2 n^{-cn^{-2}}$$

Proof of lem 3
If by induction on number of non-zero elements of $x$, Base case was 2, which we already did, (every non-zero)

Let $i$ minimize $\|x(i)\|$, and assume $\log x(i) < 0$.
Consider any $j$ s.t. $x(j) > 0$.
We have $x(j) > -x(i)$.

Set $y(k) = \begin{cases} x(i) & \text{if } i = k \\ -x(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

and set $z = x - y$.

$y$ has 2 non-zeros, and $z$ has one less than $x$. Also

$$\|x\|_1 = \|z\|_1 + \|y\|_1.$$  

So

$$\|xP^{-1}\|_1 \leq \|yP^{-1}\|_1 + \|zP^{-1}\|_1,$$

$$\leq 8 \|y\|_1 + 8 \|z\|_1 = 8 \|x\|_1,$$

where $8 = 1 - \frac{1}{n^{k-2}}$

Lemmas 4 follows from Lemma 3

by induction on $k$
Example that does not hit at time $2^{\frac{m}{2}}$:

$$ G_1 = \begin{array}{c}
3 \\
\vdots \\
1 \\
\hline
k \\
2k
\end{array} \quad \begin{array}{c}
2 \\
\hline
k+1
\end{array} \quad \begin{array}{c}
k+2 \\
\hline
k+1
\end{array} $$

Then, rotate so

$$ G_2 = \begin{array}{c}
4 \\
\vdots \\
3 \\
\vdots \\
2 \\
\hline
1 \\
k+1 \\
k+2
\end{array} \quad \begin{array}{c}
k+3 \\
\hline
k+2
\end{array} $$

and so on...