Lecture 23: The Price of Anarchy.

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I begin lecture by recalling the conditions that a Nash flow must satisfy--that the cost of every s-t path with positive flow must be the same, and the condition that the optimal solution must satisfy:

it minimizes $\sum_e f_e c_e(f_e)$, $\qquad (*)$

given that f is a flow.

Assuming that all the functions $x c_e(x)$ are convex, we see that the objective function $(*)$ has the property that every local minimum is a global minimum.

Let's see what a local minimum looks like. Recall that if the set of s-t paths is $P_1, \ldots, P_k$, and an α_i fraction of the flow travels path P_i, then the cost is

$$C(\alpha_1, \ldots \alpha_k) \overset{\text{def}}{=} \sum_i \alpha_i \, C(P_i) = \sum_i \alpha_i \sum_{e : e \in P_i} c_e(f_e) \qquad (*)$$

$$\left(\text{where } f_e = \sum_{i : e \in P_i} \alpha_i \right)$$

$$= \sum_e f_e\, C_e(f_e)$$

Let's see what happens if we shift an infinitesimal amount of flow from one path to another, say moving ε from P_1 to P_2, where $\alpha_1 > 0$.

The increase in (*) will be

$$\varepsilon \left[\frac{\partial C(\alpha_1 .. \alpha_k)}{\partial \alpha_2} - \frac{\partial C(\alpha_1 .. \alpha_k)}{\partial \alpha_1} \right]$$

as we are at a local minimum, this is ≤ 0, which says that

$$\alpha_1 > 0 \Rightarrow \frac{\partial (C(\alpha_1 .. \alpha_k))}{\partial \alpha_2} \leq \frac{\partial (C(\alpha_1 .. \alpha_k))}{\partial \alpha_1}$$

Let's see what these terms are.

Note that $\dfrac{\partial f_e}{\partial \alpha_1} = \begin{cases} 1 & \text{if } e \in P_1 \\ 0 & \text{o.w.} \end{cases}$

So,

$$\frac{\partial C(\alpha_1 .. \alpha_k)}{\partial \alpha_1} = \sum_e \frac{\partial}{\partial \alpha_1} f_e\, C_e(f_e)$$

$$= \sum \frac{\partial}{\partial \alpha_1} f_e\, C_e(f_e)$$

$$= \sum_{e \in P_1} \frac{\partial}{\partial \alpha_1} f_e \, c_e(f_e)$$

$$= \sum_{e \in P_1} c_e(f_e) + f_e \, c_e'(f_e)$$

$$= \sum_{e \in P_1} d_e(f_e),$$

where we define $d_e(x) = c_e(x) + x\, c_e'(x)$.

So, the optimum satisfies:

$\forall\, i$ s.t. $\alpha_i > 0$ and all j,

$$\sum_{e \in P_i} d_e(f_e) \le \sum_{e \in P_j} d_e(f_e) \quad (O2)$$

This is just like the conditions for Nash equilibria, but with the cost function d_e!

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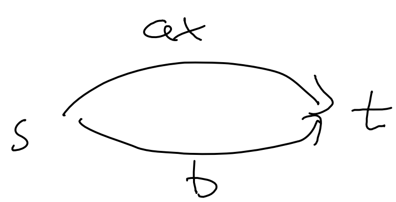
Let's now apply this analysis to see how high

$$\frac{Nash(G)}{opt(G)}$$

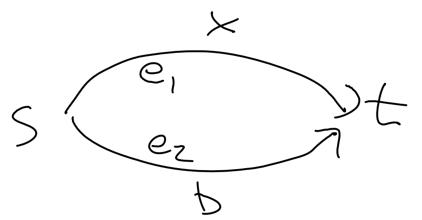
can be on a 2-link network with linear cost functions.

I assert, and will prove later, that it suffices to consider the case where one cost func has the form ax, and the other is a constant b.



As multiplying costs by a constant has no effect on $\frac{Nash}{Opt}$, we can assume $a=1$.

Also, this tells us that the analysis will be the same regardless of how much we flow, so let's assume the flow is 1



flow 1

First, let's consider the case $b \le 1$.
In this case, the Nash flow sets x so that $x = b$, and the cost of the flow is b.

For opt, compute $d_{e_1}(x) = 2x \quad d_{e_2}(x) = b$,

so it sends x over the top link satisfying

$$2x = b \implies x = \frac{b}{2}$$

The cost for opt is then

$$\frac{b}{2}\left(\frac{b}{2}\right) + \left(1 - \frac{b}{2}\right)b = b - \frac{b^2}{4} = b\left(1 - \frac{b}{4}\right)$$

So,
$$\frac{\text{Nash}}{\text{Opt}} = \frac{b}{b\left(1-\frac{b}{4}\right)} = \frac{1}{1-\frac{b}{4}} \le \frac{4}{3},$$

for $\underline{b \le 1}$

For $b \ge 1$, Nash will send all its flow over the top link, at a cost of 1.
As we increase b from 1, the cost of opt can only increase. So

$$\frac{Nash\,(b>1)}{opt\,(b>1)} \leq \frac{Nash\,(b=1)}{opt\,(b=1)} = \frac{4}{3}$$

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Bounding the Price of Anarchy.

We will now proceed to analyze networks with linear cost functions.

The outline of our analysis is as follows:
We will let f be the Nash flow in G
We will then add edges to G to get a network $\hat{G}$.
$\hat{G}$ will be based on f.
We will show that f is a Nash flow in $\hat{G}$, and we will use f to construct an optimal flow in $\hat{G}$, called g^*.

We will then show

$$\text{Nash}(\hat{G}) \le \frac{4}{3}\,\text{Opt}(\hat{G})$$

we will also have $\text{Nash}(G) = \text{Nash}(\hat{G})$

and $\text{Opt}(\hat{G}) \le \text{opt}(G)$, which

together prove

$$\text{Nash}(G) \le \frac{4}{3}\,\text{Opt}(G)$$

First, note that it suffices to consider cost functions of the form ax or b, as we can replace

$$\xrightarrow{\;ax+b\;} \quad \text{by} \quad \xrightarrow{\;ax\;}\xrightarrow{\;b\;}$$

We build $\hat{G}$ by creating, for every edge e in G, a parallel edge $\hat{e}$ of constant cost $c_e(f_e)$.

<u>Claim 1</u> The flow f is still a nash flow in $\hat{G}$

<u>pf.</u> The cost of edge e under this flow is $c_e(f_e)$ and the cost ~~of~~ edge $\hat{e}$ is also $c_e(f_e)$.
So, every s-t path in $\hat{G}$ has a length equal to the length of the path in G obtained by removing all hats, so all paths with non-zero flow have the same length

<u>Claim 2</u>
The flow $g^*_e = \frac{f_e}{2}$ $\quad g^*_{\hat{e}} = \frac{f_e}{2}$

is an optimal flow in $\hat{G}$.

pf. Note that

$$d_e(x) = 2a_e x + b_e$$

so $d_e(g_e^*) = a_e f_e + b_e = c_e(f_e)$

and $d_{\hat{e}}(x) = c_e(f_e)$, so

$$d_{\hat{e}}(g_e^*) = c_e(f_e).$$

So, under cost-function d, all s-t paths in $\hat{G}$ have the same cost under flow g_e^*, which implies g_e^*

is optimal for $\hat{G}$.

Now, let's compute the cost of f, divided by the cost of g_e^*:

$$\frac{\text{Nash}(G)}{\text{Opt}(\hat{G})} = \frac{\sum_e f_e c_e(f_e)}{\sum_e \left(\frac{f_e}{2} c_e\left(\frac{f_e}{2}\right) + \frac{f_e}{2} c_{\hat{e}}\left(\frac{f_e}{2}\right) \right)}$$

$$\leq \max_e \frac{f_e c_e(f_e)}{\frac{f_e}{2} c_e\left(\frac{f_e}{2}\right) + \frac{f_e}{2} c_{\hat{e}}\left(\frac{f_e}{2}\right)}$$

$$= \max_e \frac{c_e(f_e)}{\frac{1}{2}c_e(\frac{f_e}{2}) + \frac{1}{2}c_e(f_e)}$$

Now, if c_e is a constant, this is $\underline{1}$ whereas if $c_e = ax$, this gives

$$\frac{af_e}{\frac{1}{4}af_e + \frac{1}{2}af_e} = \frac{1}{\frac{3}{4}} = \frac{4}{3}.$$

Note that this proof technique can in general be used to reduce to the case of 2-edge graphs