Graphs and Networks Lecture 9

Resistance Distance

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9.1 Introduction

We begin where the fire alarm interrupted us in the last lecture: at Section 5 of those lecture notes.

Our main goal of this lecture is to explore the effective resistance between vertices in a graph. We will establish many fundamental properties of this quantity, and prove that it is a distance!

One thing that I need to state that I forgot to put into the lecture notes is this: if we fix the potential of $\boldsymbol{v}$ at s and t, and require that there is no external flow at any vertex other than s or t, then for all u

$$\boldsymbol{v}(s) \geq \boldsymbol{v}(u) \geq \boldsymbol{v}(t).$$

9.2 Effective Conductance

If we fix the potentials of nodes s and t to 1 and 0 respectively, and require that these are the only vertices at which current can leave or enter the circuit, the amount of flow from s to t is the effective conductance between s and t, denoted $\mathrm{C_{eff}}(s, t)$.

To see that this definition makes sense, we should show that the amount of current entering at s is equal to the amount leaving at t. Recall the equation

$$\boldsymbol{L}\boldsymbol{v} = \boldsymbol{i}_{ext}. \tag{9.1}$$

We have required that $\boldsymbol{i}_{ext}(v) = 0$ for $v \notin \{s, t\}$. As $\boldsymbol{L}$ is symmetric and $\mathbf{1}$ is in the nullspace of $\boldsymbol{L}$, every vector in the range of $\boldsymbol{L}$ is orthogonal to $\mathbf{1}$. Thus, $\boldsymbol{i}_{ext}(s) = -\boldsymbol{i}_{ext}(t)$.

9.3 Effective Resistance

As I said, we will be more interested in the effective resistance between s and t, denoted $\mathrm{R_{eff}}(s, t)$, which we first define to be the reciprocal of the effective conductance:

$$\mathrm{R_{eff}}(s, t) \stackrel{\mathrm{def}}{=} 1/\mathrm{C_{eff}}(s, t).$$

The effective resistance between s and t is equal to the potential difference we need to impose between s and t to get a current flow of 1 from s to t. To see this, just multiply equation (9.1) through by $\mathrm{R_{eff}}(s, t)$ to get

$$\boldsymbol{L}(\mathrm{R_{eff}}(s, t)\boldsymbol{v}) = \mathrm{R_{eff}}(s, t)\boldsymbol{i}_{ext}.$$

We have that

$$\mathrm{R_{eff}}(s,t)\boldsymbol{i}_{ext}(s) = 1,$$

so a current of 1 unit flows, and

$$\mathrm{R_{eff}}(s,t)\boldsymbol{v}(s) = 1 \quad \text{and} \quad \mathrm{R_{eff}}(s,t)\boldsymbol{v}(t) = 0.$$

To see why this only depends on the potential difference, not the potentials, note that the flows on edges are given by Ohm's law, which only depends on the potential differences. And,

$$\boldsymbol{L}(\boldsymbol{v} + c\mathbf{1}) = \boldsymbol{L}\boldsymbol{v},$$

for all c.

9.4 Examples

In the case of a path graph with n vertices and edges of weight 1, the effective resistance between the extreme vertices is $n-1$.

In general, if a path consists of edges of resistance $r(1,2),\ldots,r(n-1,n)$ then the effective resistance between the extreme vertices is

$$r(1,2) + \cdots + r(n-1,n).$$

To see this, set the potential of vertex i to

$$\boldsymbol{v}(i) = r(i,i+1) + \cdots + r(n-1,n).$$

Ohm's law then tells us that the current flow over the edge $(i,i+1)$ will be

$$\left(\boldsymbol{v}(i) - \boldsymbol{v}(i+1)\right)/r(i,i+1) = 1.$$

If we have k parallel edges between two nodes s and t of resistances $r_1,\ldots,r_k$, then the effective resistance is

$$\mathrm{R_{eff}}(s,t) = \frac{1}{1/r_1 + \cdots + 1/r_k}.$$

Again, to see this, note that the flow over the ith edge will be

$$\frac{1/r_i}{1/r_1 + \cdots + 1/r_k},$$

so the total flow will be 1.

9.5 Computing Effective Resistance

In general, there is no easy combinatorial way to compute effective resistance. But, we can do it with linear algebra.

To compute the effective resistance between two vertices s and t, set $\boldsymbol{x}$ to be the vector such that

$$\boldsymbol{x}(u) = \begin{cases} 1 & \text{if } u = s \\ -1 & \text{if } u = t \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\boldsymbol{x}$ is $\boldsymbol{i}_{ext}$ of the flow that injects 1 at s and removes 1 at t. So, if we solve the equation

$$\boldsymbol{L}\boldsymbol{v} = \boldsymbol{x}$$

for $\boldsymbol{v}$, we will get the potentials of this flow. As the effective resistance between s and t is the difference between the potentials at s and t, it is given by

$$\mathrm{R_{eff}}(s,t) = \boldsymbol{x}^T \boldsymbol{L}^+ \boldsymbol{x}.$$

9.6 Effective Resistance as a Distance

A distance is any function on pairs of verices such that

1. $\delta(x, x) = 0$ for every vertex x,
2. $\delta(x, y) \geq 0$ for all vertices x, y,
3. $\delta(x, y) = \delta(y, x)$, and
4. $\delta(x, z) \leq \delta(x, y) + \delta(y, x)$.

We claim that the effective resistance is a distance. The only non-trivial part to prove is the triangle inequality, (4).

Lemma 9.6.1. *Let x, y and z be vertices in a graph. Then*

$$\mathrm{R_{eff}}(x, y) + \mathrm{R_{eff}}(y, z) \geq \mathrm{R_{eff}}(x, z).$$

Proof. Let $\boldsymbol{u}$ be the potential function of the unit electrical flow from x to y in which $\boldsymbol{u}(x) = \mathrm{R_{eff}}(x, y)$ and $\boldsymbol{u}(y) = 0$. Note that for all other vertices w,

$$\boldsymbol{u}(y) \leq \boldsymbol{u}(w) \leq \boldsymbol{u}(x).$$

Similarly, let $\boldsymbol{v}$ be the potential function of the unit electrical flow from y to z in which $\boldsymbol{v}(y) = 0$ and $\boldsymbol{v}(z) = \mathrm{R_{eff}}(z)$. Again, note that for all other vertices w

$$\boldsymbol{v}(y) \leq \boldsymbol{v}(z) \leq \boldsymbol{v}(x).$$

Now, consider the potential $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$. We see that it is a flow of 1 from x to y. Moreover, the potential difference between x and y is

$$\mathrm{R}_{\mathrm{eff}}(x,z)+\boldsymbol{v}(x)-(-\mathrm{R}_{\mathrm{eff}}(z,y)+\boldsymbol{u}(z)) = \mathrm{R}_{\mathrm{eff}}(x,z)+\mathrm{R}_{\mathrm{eff}}(z,y)+\boldsymbol{v}(x)-+\boldsymbol{u}(z) \leq \mathrm{R}_{\mathrm{eff}}(x,z)+\mathrm{R}_{\mathrm{eff}}(z,y).$$

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