Graphs and Networks **out:** September 20, 2007

Problem Set 1

Lecturer: Daniel A. Spielman **due:** October 4, 2007

1 Introduction

There are 3 sections to this problem set. The first contains the 3 problems for the undergraduates. The second contains an additional problem for the graduate students. The third contains the optional experimental portion. I suggest you look at it if you are contemplating taking the experimental option on any future problem sets. The experimental portion may be substituted for 2 problems.

2 Homework Policy

You may discuss the problems with other students. But, you must write your solutions independently, drawing on your own understanding. You should cite any sources that you use on the problem sets other than the textbook, TA and instructor. This means that you should list your collaborators.

You **may not** search the web for solutions to similar problems given out in other classes. If you think this policy needs any clarification, please let me know.

3 Undergraduate Problems

1. Let $G = (V, E)$ be a graph, and consider the lazy random walk on G. For vertices a and b, let $\boldsymbol{p}_t^a(b)$ denote the probability that the lazy random walk starting at a is at b after t time steps. (If you prefer to consider the ordinary random walk instead of the lazy random walk, that is fine too.)

   a. Prove that if G is regular, then
   $$\boldsymbol{p}_t^a(b) = \boldsymbol{p}_t^b(a).$$

   b. Prove that for a general undirected graph G,
   $$d(a)\boldsymbol{p}_t^a(b) = d(b)\boldsymbol{p}_t^b(a).$$

2. Recall that in Lecture 2 we defined $r(G)$ to be the assortativity of the graph G.

   (a) Find a graph with assortativity -1, or at least construct an infinite family of graphs in which some graph has assortativity less than $-1 + \epsilon$ for every $\epsilon > 0$.

(b) Find a graph with assortativity 1, or at least construct an infinite family of graphs in which some graph has assortativity at least $1-\epsilon$ for every $\epsilon > 0$.

3. In the following problems, $G = (V, E)$ is an undirected, connected graph, and W is its lazy walk matrix.

   (a) Prove that if G is bipartite, then W is singular. It is easiest to show this by constructing a vector $\boldsymbol{x}$ satisfying
   $$\boldsymbol{W}\boldsymbol{x} = \boldsymbol{0} \qquad \text{or} \qquad \boldsymbol{x}\,\boldsymbol{W} = \boldsymbol{0}.$$
   Hint: If you are stuck, first consider the example of the graph with two vertices and one edge.

   (b) Prove that if G is connected and there exists a vector $\boldsymbol{x}$ satisfying $\boldsymbol{x}\,\boldsymbol{W} = \boldsymbol{0}$, then G is bipartite. Hint: Modify consider the proof of Lemma 4.2.1 from Lecture 4.

   (c) Prove that if G is connected and bipartite, then for every eigenvalue λ of $\boldsymbol{W}$, $1-\lambda$ is also an eigenvalue of $\boldsymbol{W}$.

4 Graduate Problem

4. The Perron-Frobenius theorem tell us that if A is the adjacency matrix of a weighted, strongly connected, directed graph, then A has a strictly positive eigenvector v with positive eigenvalue μ_1. Moreover, all other eigenvalues μ_i of A satisfy $|\mu_i| < \mu_1$. It turns out that this theorem is easy to prove in the case that A is stochastic, that is $A\mathbf{1} = \mathbf{1}$.

   In this problem, we will show how to reduce to the case that $A\mathbf{1} = c\mathbf{1}$, at least when A has no zero entries. That is, we will prove that every such A is similar to a constant times a stochastic matrix.

   We will do this algorithmically. Let $A_0 = A$. Then, set $s^{(i)} = A_i\mathbf{1}$, and $D_i = \mathtt{diag}(s^{(i)})$. We then set $A_{i+1} = D_i^{-1}A_iD_i$. We will show that the sequence of matrices A_i is converging to a constant times a stochastic matrix.

   (a) Let $s^{(i)}_{max} = \max_i s^{(i)}$ denote the maximum row-sum in A_i, and $s^{(i)}_{min}$ denote the minimum row sum. Prove that
   $$s^{i+1}_{max} - s^{i+1}_{min} \leq s^{i}_{max} - s^{i}_{min}.$$

   (b) For a matrix A, let $\min(A)$ denote the minimum entry of A. Set
   $$\gamma_i = \frac{\min(A_i)}{s^{(i)}_{max}}.$$
   Prove that
   $$s^{i+1}_{max} - s^{i+1}_{min} \leq (1-\gamma_i)\left(s^{i}_{max} - s^{i}_{min}\right).$$

   (c) Prove that for every matrix A, there exists a constant ϵ such that $\gamma_i \geq \epsilon$, for all i.

Taken together, these statements show that if A has no zero entries, then the sequence A_i approaches a multiple of a stochastic matrix. With a little analysis, one can extend this to show that A is similar to a multiple of a stochastic matrix.

(extra credit) Extend this analysis to the case in which A can have zero entries, but is the weighted adjacency matrix of a strongly connected directed graph.

5 Experimental Problem

This problem can be summarized as "get a graph and tell me a little about it". To be more specific, you should find a graph that you are going to use in experimental work throughout the class (although you can switch graphs later if you become unhappy with your original choice). The graph should be from some "real-world" source. That is, it should not be constructed at random, or come from an algebraic construction. The graph should have at least 1,000 nodes, and preferably at least 10,000.

The best way to get a graph is probably to write a script to read it or grab it from the web. Interesting sources include databases of papers (arxiv, ncstrl), fragements of the web (wikipedia, links interal to a university, etc.). Wget is a useful tool for downloading a large portion of the web, but it is overkill since you only need the links. This part could take a lot of work.

Here's what you should report:

a. What is your graph, and where did you get it.

b. How are you storing your graph?

c. How many nodes are there in your graph?

d. How many edges does it have?

e. How many nodes does it have of degree 1?

f. How many nodes does it have of degree 2?

Submit all code you've used.